The truncated Fourier operator. VI.

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Mathematics Subject Classification: (2000). 35S30, 43A90.

Keywords: Truncated Fourier operator, spectral analysis.

Abstract

The spectral analysis of the operator Fourier truncated on the positive half-axis is done.

6 Spectral theory of the Fourier operator truncated on the positive half-axis.

In this section we study the truncated Fourier operator $\mathcal{F}_E$,

$$(\mathcal{F}_E x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_E x(\xi)e^{it\xi}\, d\xi\,, \quad t \in E, \tag{6.1}$$

in the case when the set E is the positive half-axis: $E = [0, \infty)$. The operator $\mathcal{F}_E$ is considered as an operator acting in the space $L^2(E)$ of all square measurable complex valued functions on E provided with the scalar product

$$\langle x, y \rangle = \int\limits_E x(t)\overline{y(t)}\, dt\,.$$

The operator $\mathcal{F}_E^*$ adjoint to the operator $\mathcal{F}_E$ with respect to this scalar product is

$$(\mathcal{F}_E^* x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_E x(\xi)e^{-it\xi}\, d\xi\,, \quad t \in E\,. \tag{6.2}$$

This set E is not symmetric and bounded from below. According to [KaMa1, Theorem 1.4], the operator $\mathcal{F}_E, E = [0,\infty)$, is not a normal operator. However we can do without Theorem 1.4. The fact that the operator $\mathcal{F}_E, E = [0,\infty)$ is not normal will be evident after we study its spectral properties in more detail.

1. As in the previous consideration, related to the non-truncated Fourier operator, (corresponding to the set $E = (-\infty,\infty)$), or to the Fourier operator truncated on the finite symmetric interval, (corresponding to the set $E = (-a,a)$, $0 < a < \infty$), also in the case when $E = (0,\infty)$ our reasoning is based on using some selfadjoint differential operator $\mathcal{L}$ which commutes with the operator $\mathcal{F}_E$.

The *formal* differential operator L which generate this operator $\mathcal{L}$ is

$$(Lx)(t) = -\frac{d}{dt}\left(t^2\frac{dx(t)}{dt}\right) \tag{6.3}$$

The formal operator L generates the minimal operator $\mathcal{L}_{\min}$ and the maximal operator $\mathcal{L}_{\max}$.

The formal operator L describes how act the operators $\mathcal{L}_{\min}$ and $\mathcal{L}_{\max}$ on functions from the appropriate domain of definition.

Definition 6.1. *The set $\mathcal{A}$ is the set of complex valued functions $x(t)$ defined on the open half-axis $(0,\infty)$ and satisfying the following conditions:*

1. *The derivative $\dfrac{dx(t)}{dt}$ of the function $x(t)$ exists at every point t of the interval $(0,\infty)$;*
2. *The function $\dfrac{dx(t)}{dt}$ is absolutely continuous on every compact subinterval of the interval $(0,\infty)$;*

Definition 6.2. *The set $\mathring{\mathcal{A}}$ is the set of complex-valued functions $x(t)$ defined on the open interval $(0,\infty)$ and satisfied the following conditions:*

1. *The function $x(t)$ belongs to the set $\mathcal{A}$ defined above;*
2. *The support $\operatorname{supp} x$ of the function $x(t)$ is a compact subset of the open interval $(0,\infty)$: $(\operatorname{supp} x) \Subset (-a,a)$.*

Definition 6.3. *The differential operator $\mathcal{L}_{\max}$ is defined as follows:*

1. *The domain of definition* $\mathcal{D}_{\mathcal{L}_{\max}}$ *of the operator* $\mathcal{L}_{\max}$ *is:*

$$\mathcal{D}_{\mathcal{L}_{\max}} = \{x : \, x(t) \in L^2((0,\infty)) \cap \mathcal{A} \ \text{ and } \ (Lx)(t) \in L^2((0,\infty))\}, \tag{6.4a}$$

where $(Lx)(t)$ *is defined*[1] *by* (6.3).

2. *The action of the operator* $\mathcal{L}_{\max}$ *is:*

$$\text{For } x \in \mathcal{D}_{\mathcal{L}_{\max}}, \quad \mathcal{L}_{\max} x = Lx\,. \tag{6.4b}$$

The operator $\mathcal{L}_{\max}$ *is said to be the* maximal differential operator generated by the formal differential expression L.

The minimal differential operator $\mathcal{L}_{\min}$ is the restriction of the maximal differential operator $\mathcal{L}_{\max}$ on the set of functions which is some sense vanish at the endpoint of the interval $(0,\infty)$. The precise definition is presented below.

Definition 6.4. *The operator* $\mathcal{L}_{\min}$ *is the closure*[2] *of the operator* $\mathring{\mathcal{L}}$*:*

$$\mathcal{L}_{\min} = \operatorname{clos}\big(\mathring{\mathcal{L}}\,\big)\,, \tag{6.5a}$$

where the operator $\mathring{\mathcal{L}}$ *is the restriction of the operator* $\mathcal{L}_{\max}$*:*

$$\mathring{\mathcal{L}} \subset \mathcal{L}_{\max}, \quad \mathring{\mathcal{L}} = \mathcal{L}_{\max}|_{\mathcal{D}_{\mathring{\mathcal{L}}}}, \quad \mathcal{D}_{\mathring{\mathcal{L}}} = \mathcal{D}_{\mathcal{L}_{\max}} \cap \mathring{\mathcal{A}}\,. \tag{6.5b}$$

By $\langle\,,\,\rangle$ we denote the standard scalar product in $L^2((0,\infty))$:

$$\text{For } u,\, v \in L^2((0,\infty)), \quad \langle u, v\rangle = \int\limits_0^\infty u(t)\overline{v(t)}\,dt\,.$$

The properties of the operators $\mathcal{L}_{\min}$ and $\mathcal{L}_{\max}$:

1. *The operator* $\mathcal{L}_{\min}$ *is symmetric:*

$$\langle \mathcal{L}_{\min} x, y\rangle = \langle x, \mathcal{L}_{\min} y\rangle\,, \quad \forall x,\, y \in \mathcal{D}_{\mathcal{L}_{\min}}; \tag{6.6}$$

In other words, the operator $\mathcal{L}_{\min}$ *is contained in its adjoint*:

$$\mathcal{L}_{\min} \subseteq (\mathcal{L}_{\min})^*\,;$$

[1] Since $x \in \mathcal{A}$, the expression $(Lx)(t)$ is well defined.

[2] Since the operator $\mathring{\mathcal{L}}$ is symmetric, it is closable.

2. *The operators* $\mathcal{L}_{\min}$ *and* $\mathcal{L}_{\max}$ *are mutually adjoint*:

$$(\mathring{\mathcal{L}})^* = (\mathcal{L}_{\min})^* = \mathcal{L}_{\max}, \quad (\mathcal{L}_{\max})^* = \mathcal{L}_{\min}\,; \tag{6.7}$$

The fact that $(\mathring{\mathcal{L}})^* = \mathcal{L}_{\max}$ is a very general fact related to ordinary differential operators, regular or singular, of finite or infinite interval.

Let as calculate the deficiency indices of the symmetric operator $\mathcal{L}_{\min}$. In view of (6.7), we have to investigate the dimension of the space of solutions of the equation $\mathcal{L}_{\max}x = \lambda x$ for λ from the upper half plane and for λ from the lower half plane. The equation $\mathcal{L}_{\max}x = \lambda x$ is the differential equation of the form

$$-\frac{d}{dt}\left(t^2\frac{dx(t)}{dt}\right) = \lambda x(t)\,, \quad 0 < t < \infty\,. \tag{6.8}$$

We are interested in solutions of this equation which belong to $L^2(0,\infty)$.

The equation (6.8) can be solved explicitly. Seeking its solution on the form $x(t) = t^a$, we come to the equation

$$a(a+1) + \lambda = 0\,.$$

The roots of this equation are

$$a_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda}, \quad a_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda}\,. \tag{6.9}$$

These roots are different if $\lambda \neq \frac{1}{4}$. Thus if $\lambda \neq \frac{1}{4}$, the general solution of the differential equation (6.8) is of the form

$$x(t) = c_1 t^{a_1} + c_2 t^{a_2}\,, \tag{6.10}$$

where c_1, c_2 are arbitrary constants. If $\lambda = \frac{1}{4}$, the general solution of (6.8) is of the form

$$x(t) = c_1 t^{1/2} + c_2 t^{1/2}\ln t\,. \tag{6.11}$$

However the function $x(t)$ of the form (6.10) (or (6.11)) belongs to $L^2((0,\infty))$ only if $x(t) \equiv 0$. Thus, the following result is proved

Lemma 6.1. *Whatever* $\lambda \in \mathbb{C}$ *is, the differential equation* (6.8) *has no solutions* $x(t) \not\equiv 0$ *belonging to* $L^2((0,\infty))$.

In particular, taking $\lambda = i$ and $\lambda = -i$, we see that the deficiency indices n_+ and n_- of the symmetric operator $\mathcal{L}_{\min}$ are equal to zero. Applying the von Neumann criterion of the selfadjointness, we obtain

Lemma 6.2. *The differential operator $\mathcal{L}_{\min}$ is self-adjoint.*

In other words, we prove that $\mathcal{L}_{\min} = \mathcal{L}_{\max}$.

Notation 6.1. *From now till the end this paper we use the notation $\mathcal{L}$ for the operator $\mathcal{L}_{\min} = \mathcal{L}_{\max}$:*

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}_{\min} = \mathcal{L}_{\max} \tag{6.12}$$

Since $\mathcal{L} = \mathcal{L}_{\min}$,

$$\mathcal{L} = \operatorname{clos} \mathring{\mathcal{L}} . \tag{6.13}$$

Since $\mathcal{L} = \mathcal{L}_{\max}$,

$$\mathcal{D}_{\mathcal{L}} = \left\{ x : \ x \in \mathcal{A} \cap L^2(0,\infty), \ Lx \in L^2(0,\infty) \right\} . \tag{6.14}$$

Theorem 6.1. *The (selfadjoint) operator $\mathcal{L}$ commutes with the truncated Fourier operator $\mathcal{F}_E$, $E = (0,\infty)$ as well as with the adjoint operator $\mathcal{F}_E^*$:*

1. *If $x \in \mathcal{D}_{\mathcal{L}}$, then $\mathcal{F}_E\, x \in \mathcal{D}_{\mathcal{L}}$, $\mathcal{F}_E^*\, x \in \mathcal{D}_{\mathcal{L}}$.*
2. $$\mathcal{F}_E \mathcal{L}\, x = \mathcal{L} \mathcal{F}_E\, x, \quad \mathcal{F}_E^* \mathcal{L}\, x = \mathcal{L} \mathcal{F}_E^*\, x, \quad \forall\, x \in \mathcal{D}_{\mathcal{L}} . \tag{6.15}$$
3. *If $\mathcal{E}(\Delta)$ is the spectral projector for the operator $\mathcal{L}$ corresponding to a Borelian subset Δ of the real axis, then*

$$\mathcal{F}_E \mathcal{E}(\Delta) = \mathcal{E}(\Delta) \mathcal{F}_E, \quad \mathcal{F}_E^* \mathcal{E}(\Delta) = \mathcal{E}(\Delta) \mathcal{F}_E^* \quad \forall\, \Delta . \tag{6.16}$$

Proof. The proof is based on Lemma 2.3 and Theorem 2.4 from [KaMa2]. Let $x \in \mathcal{D}_{\mathring{\mathcal{L}}}$. Then the function $\mathcal{F}_E\, x$ is the Fourier transform of a summable finite function, hence $\mathcal{F}_E\, x \in \mathcal{A}$. Since $\mathcal{F}_E L^2((0,\infty)) \subseteq L^2((0,\infty))$, and $x \in L^2((0,\infty))$, $Lx \in L^2((0,\infty))$, then $\mathcal{F}_E x \in L^2((0,\infty))$, $\mathcal{F}_E Lx \in L^2((0,\infty))$. Thus,

$$\mathcal{F}_E \mathcal{D}_{\mathring{\mathcal{L}}} \subseteq \mathcal{D}_{\mathcal{L}} \tag{6.17}$$

Since the support of the function $x(t)$ is a compact subset of the open interval $(0,\infty)$, the condition (2.43) is satisfied. According to Theorem 2.4 from [KaMa2], the equality

$$\mathcal{F}_E \mathcal{L}\, x = \mathcal{L} \mathcal{F}_E\, x, \quad \forall x \in \mathcal{D}_{\mathring{\mathcal{L}}} . \tag{6.18}$$

holds.

Let $x \in \mathcal{D}_{\mathcal{L}}$ now. In view of (6.13), there exists a sequence $x_n \in \mathcal{D}_{\mathring{\mathcal{L}}}$ such that $x_n \to x$, $\mathcal{L}x_n \to \mathcal{L}x$ as $n \to \infty$. (The convergence is the

strong convergence, that is the convergence in $L^2((0,\infty))$.) According to (6.18), for every n the equality

$$\mathscr{F}_E \mathcal{L}\, x_n = \mathcal{L} \mathscr{F}_E\, x_n \tag{6.19}$$

holds. The operator $\mathscr{F}_E$ is continuous. Terefore $\mathscr{F}_E\, x_n \to \mathscr{F}_E\, x$, and $\mathscr{F}_E\, \mathcal{L}x_n \to \mathscr{F}_E\, \mathcal{L}x$ as $n \to \infty$. Now from (6.19) it follows that there exists limit of the sequence $\mathcal{L}(\mathscr{F}_E\, x_n)$. Since the operator $\mathcal{L}$ is closed, then $\mathscr{F}_E\, x \in \mathcal{D}_{\mathcal{L}}$, and $\mathscr{F}_E \mathcal{L}\, x = \mathcal{L}\mathscr{F}_E\, x$. The inclusion $\mathscr{F}_E\, x \in \mathcal{D}_{\mathcal{L}}$ and the equality $\mathscr{F}_E^{*} \mathcal{L}\, x = \mathcal{L}\mathscr{F}_E^{*}\, x$ can be established analogously.

Since the operator $\mathcal{L}$ is selfadjoint, its spectrum is real. In particular, for every non-real number z, the operator $\mathcal{L} - zI$ is invertible, and its inverse operator $(\mathcal{L} - zI)^{-1}$ is bounded and defined everywhere. Taking $x = (\mathcal{L} - zI)^{-1}y$ in (6.15), where y is an arbitrary vector, we obtain that

$$(\mathcal{L} - zI)^{-1}\mathcal{F}_E = \mathcal{F}_E(\mathcal{L} - zI)^{-1} \quad \forall\, z:\ \operatorname{Re} z \neq 0\,. \tag{6.20}$$

The equality (6.16) is a consequence of (6.20). □

2. Let as investigate the spectral structure of the operator $\mathcal{L}$. We shall see that the properties of the operator $\mathcal{L}$ corresponding to the case $E = (0,\infty)$ differs from the properties of the operators $\mathcal{L}$ corresponding to the cases $E = (-\infty,\infty)$ and $E = (-a,a)$, $0 < a < \infty$. The spectrum of the differential operator $\mathcal{L}$ corresponding to $E = (0,\infty)$ is continuous and of multiplicity two, whereas the spectra of the differential operators $\mathcal{L}$ corresponding to the cases $E = (-\infty,\infty)$ and $E = (-a,a)$ are discrete and of multiplicity one.

Fortunately, the spectral analysis of the operator $\mathcal{L}$ can be reduced to the spectral analysis of the operator $-\frac{d^2}{ds^2}$ in $L^2((-\infty,\infty))$. Changing variables

$$t = e^s,\ -\infty < s < \infty, \quad y(s) = e^{s/2}x(e^s)\,, \tag{6.21}$$

we reduce the equation (6.8) to the form

$$-\frac{d^2y(s)}{ds^2} + \frac{1}{4}y(s) = \lambda\, y(s), \quad -\infty < s < \infty\,. \tag{6.22}$$

The correspondence

$$y = Ux, \quad \text{where } \ y(s) = e^{s/2}x(e^s)\,, \tag{6.23}$$

is an unitary operator from $L^2((0,\infty),\,dt)$ onto $L^2((-\infty,\infty),\,ds)$:

$$\int\limits_0^\infty |x(t)|^2\,dt = \int\limits_{-\infty}^\infty |y(s)|^2 ds\,. \tag{6.24}$$

The operator $\mathcal{L}$ is unitarily equivalent to the operator $\mathcal{T}+\frac{1}{4}I$:

$$\mathcal{L} = U^{-1}\Big(\mathcal{T}+\frac{1}{4}I\Big)U, \tag{6.25}$$

where

$$(\mathcal{T}y)(s) = -\frac{d^2y(s)}{ds^2} \tag{6.26}$$

is the differential operator in $L^2(-\infty,\infty)$ defined on the "natural" domain. The spectral structure of the operator $\mathcal{T}$ is well known. Its spectrum $\sigma_{\mathcal{T}}$ is absolutely continuous of multiplicity two and fills the positive half-axis: $\sigma_{\mathcal{T}} = [0,\infty)$. The (generalized) eigenfunctions of the operator $\mathcal{T}$ corresponding to the point $\rho \in (0,\infty)$ are

$$e_+(s,\mu) = e^{i\mu s}\,,\quad e_-(s,\mu) = e^{-i\mu s},\quad -\infty < s < \infty\,, \tag{6.27}$$

where $\mu = \rho^{1/2} > 0$. Changing variable in the expressions (6.27) for eigenfunctions of the operator $\mathcal{T}$ according to (6.21), we come to the functions

$$\psi_+(t) = t^{-\frac{1}{2}+i\mu},\quad \psi_-(t) = t^{-\frac{1}{2}-i\mu},\quad t\in(0,\infty),\ 0<\mu<\infty\,. \tag{6.28}$$

Both of the functions $\psi_+(t,\mu), \psi_-(t,\mu)$ are eigenfunctions of the operator $\mathcal{L}$ corresponding to *the same* eigenvalue $\lambda(\mu)$,

$$\lambda(\mu) = \mu^2 + 1/4,\quad 0<\mu<\infty\,. \tag{6.29}$$

$$\mathcal{L}\psi_+(t,\mu) = \lambda(\mu)\psi_+(t,\mu),\quad \mathcal{L}\psi_-(t,\mu) = \lambda(\mu)\psi_-(t,\mu)\,. \tag{6.30}$$

In view of (6.25), the spectral properties of the operator $\mathcal{T}$ can be reformulated as the spectral properties of the operator $\mathcal{L}$. Reindexing the spectral parameter μ in such a manner that the value of the parameter to be coincide with the eigenvalue, we come to the functions

$$\varphi_+(t,\lambda) = \psi_-(t,\mu(\lambda)),\quad \varphi_-(t,\lambda) = \psi_-(t,\mu(\lambda)). \tag{6.31}$$

where

$$\mu = \mu(\lambda) = \sqrt{\lambda - \frac{1}{4}},\quad \mu>0\,,\quad 1/4<\lambda<\infty\,. \tag{6.32}$$

$$\mathcal{L}\varphi_+(t,\lambda) = \lambda\varphi_+(t,\lambda), \quad \mathcal{L}\varphi_2- = \lambda\varphi_-(t,\lambda)\,, \quad 1/4 < \lambda < \infty\,. \tag{6.33}$$

In what follow we work mainly with the system

$$\{\psi_+(t,\mu), \psi_-(t,\mu)\}_{\mu\in(0,\infty)}$$

of "non-reindexed" eigenfunctions, but not with the system

$$\{\varphi_+(t,\lambda), \varphi_-(t,\lambda)\}_{\lambda\in(1/4,\infty)}$$

of "reindexed" eigenfunctions. The reindexing procedure is useful if we would like to feet the eigenfunctions to the operator $\mathcal{L}$ in a most natural way. However, the operator $\mathcal{L}$ plays the heuristic role only. What we actually need these are eigenfunctions of $\mathcal{L}$ but not $\mathcal{L}$ itself.

The spectrum $\sigma_{\mathcal{L}}$ of the operator $\mathcal{L}$ is absolutely continuous of multiplicity two and fills of the semi-infinite interval: $\sigma_{\mathcal{L}} = [\frac{1}{4}, \infty)$. To the point $\lambda \in (\frac{1}{4}, \infty)$ of the spectrum of the operator $\mathcal{L}$ corresponds the two-dimensional "generalized eigenspace" generated by the 'generalized" eigenfunctions $\psi_+(t,\mu(\lambda))$, $\psi_-(t,\mu(\lambda))$, $\mu(\lambda)$ is defined in (6.32).

Given $\mu \in (0,\infty)$, the "eigenfunctions" $\psi_+(t,\,\mu)$, $\psi_-(t,\,\mu)$ do not belong to the space $L^2((0,\infty),\,dt)$, but almost belong. Their averages with respect to the spectral parameter

$$\frac{1}{2\varepsilon}\int\limits_{\mu-\varepsilon}^{\mu+\varepsilon} \psi_\pm(t,\zeta)\,d\zeta = t^{-\frac{1}{2}\pm i\mu}\frac{\sin(\varepsilon\ln t)}{\varepsilon\ln t}$$

over an arbitrary small interval $(\mu-\varepsilon,\mu+\varepsilon) \subset (0,\,\infty)$ already belongs to $L^2((0,\infty),\,dt)$. These eigenfunctions satisfy the generalized "orthogonality relations":

$$\int\limits_0^\infty \psi_+(t,\mu_1)\,\overline{\psi_-(t,\mu_2)}\,dt = 0 \quad ,$$

$$\int\limits_0^\infty \psi_+(t,\mu_1)\,\overline{\psi_+(t,\mu_2)}\,dt = 2\pi\,\delta(\mu_1-\mu_2),$$

$$\int\limits_0^\infty \psi_-(t,\mu_1)\,\overline{\psi_-(t,\mu_2)}\,dt = 2\pi\,\delta(\mu_1-\mu_2),$$

$$\forall\,\mu_1,\mu_2 > 0, \quad \text{where} \quad \delta \quad \text{is the Dirac } \delta\text{-function.} \tag{6.34}$$

The integrals in (6.34) diverge, so the relations (6.34) are nonsense if they are understood literally. Nevertheless the equalities (6.34) can be provide with a meaning in the sense of distributions.

However we prefer to stay on the 'classical' point of view, and to to formulate the 'orthogonality properties' of the 'eigenfunctions' $\psi_{\pm}(t,\lambda)$ in the language of the L^2-theory of the Fourier integrals.

Notation. In what follows we use the matrix notation, matrix language, and matrix operations. We organize the pair $\psi_+(t,\mu)$, $\psi_-(t,\mu)$, (6.28), into the matrix-column

$$\psi(t,\mu) = \begin{bmatrix} \psi_+(t,\mu) \\ \psi_-(t,\mu) \end{bmatrix} . \tag{6.35a}$$

According to the matrix algebra notation, the matrix adjoint to the matrix-column $\psi(t,\mu)$ is the matrix-row

$$\psi^*(t,\mu) = \begin{bmatrix} \overline{\psi_+(t,\mu)} & \overline{\psi_-(t,\mu)} \end{bmatrix} . \tag{6.35b}$$

In this notation, the orthogonality relation (6.34) can be presented as

$$\int\limits_{(0,\infty)} \psi(t,\mu_1)\psi^*(t,\mu_2)\,dt = 2\pi\,\delta(\mu_1-\mu_2)I, \tag{6.36}$$

where I is the 2×2 identity matrix.

Theorem 6.2. *Given a function $x(t) \in L^2((0,\infty),dt)$, let us define its "Fourier transform" with respect to the "orthogonal system"* (6.28) *as the function*[3] $\hat{x}(\mu) = \begin{bmatrix} \hat{x_+}(\mu) & \hat{x}_-(\mu) \end{bmatrix}$:

$$\hat{x}(\mu) = \int\limits_{(0,\infty)} x(t)\psi^*(t,\mu)\,dt\,, \quad \mu \in (0,\infty)\,. \tag{6.37}$$

Then

1. *The functions $\hat{x}(\mu)$ belong to $L^2((0,\infty),d\mu)\oplus L^2((0,\infty),d\mu)$, and the Parseval identity holds:*

$$\int\limits_{(0,\infty)} |x(t)|^2\,dt = \int\limits_{(0,\infty)} \hat{x}(\mu)\,\hat{x}^*(\mu)\,\frac{d\mu}{2\pi}\,\cdot \tag{6.38}$$

[3] The values $\hat{x}(\mu)$ are 1×2 vector-rows.

2. *The function $x(t)$ can be recovered from its "Fourier transform" $\hat{x}(\mu)$ as the "inverse Fourier transform":*

$$x(t) = \int\limits_{(0,\infty)} \hat{x}(\mu)\, \psi(t,\mu)\, \frac{d\mu}{2\pi}\,. \tag{6.39}$$

3. *Given a preassigned 1×2 row-function $\tilde{x}(\mu)$ which is square summable on $(0,\infty)$: $\int\limits_{(0,\infty)} \tilde{x}(\mu)\tilde{x}^*(\mu)\frac{d\mu}{2\pi} < \infty$, there exists the function $x(t) \in L^2((0,\infty),\, dt)$ whose "Fourier transform" $\hat{x}(\mu)$ coincide with $\tilde{x}(\mu)$:*

$$\hat{x}(\mu) = \tilde{x}(\mu)\,.$$

This function $x(t)$ can be constructed from $\tilde{x}(\mu)$ by the "inverse Fourier transform" :

$$x(t) \stackrel{\text{def}}{=} \int\limits_{\mu\in(0,\infty)} \tilde{x}(\mu)\, \psi(t,\mu)\, \frac{d\mu}{2\pi}\,. \tag{6.40}$$

4. *A function $x(t) \in L^2((0,\infty),\, dt)$ belongs to the domain of definition $\mathcal{D}_{\mathcal{L}}$ of the operator $\mathcal{L}$, (6.12), if and only if the condition*

$$\int\limits_{(0,\,\infty)} \lambda(\mu)^2\, \hat{x}(\mu)\, \hat{x}^*(\mu) \frac{d\mu}{2\pi} < \infty \tag{6.41}$$

is satisfied.

5. *If $x(t) \in \mathcal{D}_{\mathcal{L}}$, then the "Fourier transform" $\hat{x}(\mu)$ of $x(t)$, defined by (6.37), are related with the "Fourier transform" $(\widehat{\mathcal{L}x})(\mu)$ of the function $(\mathcal{L}x)(t)$:*

$$(\widehat{\mathcal{L}x})(\mu) = \int\limits_0^\infty (\mathcal{L}x)(t)\psi^*(t,\mu)\, dt\,, \tag{6.42}$$

by the equalitity

$$(\widehat{\mathcal{L}x})(\mu) = \lambda(\mu)\hat{x}(\mu)\,. \tag{6.43}$$

In other words, if the expansion of the function $x(t)$ is the form (6.39), then the expansion of the function $(\mathcal{L}x)(t)$ is of the form

$$(\mathcal{L}x)(t) = \int\limits_{\mu\in(0,\infty)} \lambda(\mu)\hat{x}(\mu)\, \psi(t,\mu)\, \frac{d\mu}{2\pi}\cdot \tag{6.44}$$

6. *If z is an arbitrary non-real number, then the resolvent $(\mathcal{L}-zI)^{-1}$ of the operator $\mathcal{L}$ is expressible as*

$$(\mathcal{L}-zI)^{-1}x(t) = \int\limits_{\mu\in(0,\infty)} \frac{1}{\lambda(\mu)-z}\,\hat{x}(\mu)\,\psi(t,\mu)\,\frac{d\mu}{2\pi}\,. \tag{6.45}$$

7. *The spectral projector $\mathcal{E}(\Delta)$ for the operator $\mathcal{L}$ corresponding to a Borelian subset Δ of the real axis is expressible as*

$$(\mathcal{E}(\Delta)x)(t) = \int\limits_{\mu\in(0,\infty)} \chi_\Delta(\lambda(\mu))\,\hat{x}(\mu)\,\psi(t,\mu)\,\frac{d\mu}{2\pi}\,, \tag{6.46}$$

where $\chi_\Delta(\lambda)=1$ if $\lambda\in\Delta$, $\chi_\Delta(\lambda)=0$ if $\lambda\notin\Delta$.

Remark 6.1. *The transformation* (6.37)*: $x(t)\to\hat{x}(\mu)$ is defined like it is usually made in the L^2-theory of Fourier integral. This transformation is firstly defined for functions $x(t)\in L^2$ having compact support in $(0,\infty)$. Such x belong to L^1. So the integrals in* (6.37) *are defined in the proper sense, as Lebesgue integrals. Moreover the Parseval equality* (6.38) *holds for such $x(t)$. Thus the transformation* (6.37)*: $x(t)\to\hat{x}(\mu)$ is an isometric transformation from $L^2(dt)$ into $L^2(d\mu)$ which is well defined on the set of x dense in $L^2(dt)$. Then this isometric transformation is extended from this set to the whole $L^2(dt)$ by the continuity. The inverse transformation* (6.40)*: $\tilde{x}(\mu)\to x(t)$, acting from $L^2(d\mu)$ into $L^2(dt)$, is defined and considered analogously.*

Proof.
∘ The assertions $1,2$ and 3 of Theorem are the main facts of the L^2-theory of the Fourier integral, which are reformulated in the way suitable for application to the spectral theory of the operator $\mathcal{L}$.

Given the function $y(s)\in L^2((-\infty,\infty),ds)$, its Fourier transform $\tilde{y}(\mu)$ is

$$\tilde{y}(\mu) = \int\limits_{(-\infty,\infty)} y(s)e^{-i\mu s}ds\,.$$

We split the function $\tilde{y}(\mu)$ into the pair $\tilde{y}_1(\mu)$ and $\tilde{y}_2(\mu)$, both functions $\tilde{y}_1(\mu)$ and $\tilde{y}_2(\mu)$ are defined for $\mu>0$:

$$\tilde{y}_+(\mu) = \int\limits_{(-\infty,\infty)} y(s)e^{-i\mu s}, \quad \tilde{y}_-(\mu) = \int\limits_{(-\infty,\infty)} y(s)e^{i\mu s}, \quad \mu>0\,. \tag{6.47}$$

The Parseval identity and the inversion formula can by presented in the form

$$\int\limits_{(-\infty,\infty)} |y(s)|^2 ds = \int\limits_{(0,\infty)} |\tilde{y}_+(\mu)|^2 \frac{d\mu}{2\pi} + \int\limits_{(0,\infty)} |\tilde{y}_-(\mu)|^2 \frac{d\mu}{2\pi}\,, \tag{6.48}$$

and

$$y(s) = \int\limits_{(0,\infty)} \tilde{y}_+(\mu)\, e^{i\mu s} \frac{d\mu}{2\pi} + \int\limits_{(0,\infty)} \tilde{y}_-(\mu)\, e^{-i\mu s} \frac{d\mu}{2\pi}\,. \tag{6.49}$$

Changing variable

$$x(t) = t^{-1/2} y(\ln t), \ \ \mu = \sqrt{\lambda - 1/4}\,,$$

(see (6.21)), we present the formulas (6.47), (6.48) and (6.49) in the form (6.37), (6.38) and (6.39) respectively.
∘ Assertion 4 is the reformulation of the condition for the second derivative of a function to be square summable.
∘ Assertions 5 is the reformulation of the rule how to express the Fourier transform of the second derivative of a function in terms of the Fourier transform of the function itself.
∘ Assertion 7 is the consequence of assertion 6. □

We do not present the proof of the assertions 4-7 of the theorem in detail. This theorem plays an heuristic role only. The only what we need are the expressions (6.28) for the generalized eigenfunctions of $\mathcal{L}$ corresponding to the point $\mu(\lambda)$ of the spectrum of $\mathcal{L}$.

3. Now we obtain the representation of the truncated Fourier operator $\mathcal{F}_E$, $E = (0,\infty)$ in the form of an expansion over the eigenfunctions of the operator $\mathcal{L}$. First we do formal calculations. Then we justify them.

The operator $\mathcal{L}$ commutes with the operators $\mathcal{F}_E$, $\mathcal{F}_E^*$, $E = [0,\infty)$. (Theorem 6.1). Let $\mu > 0$. The "eigenspace" of the operator $\mathcal{L}$ corresponding to the eigenvalue $\lambda(\mu)$ is two-dimensional and is generated by the "eigenfunctions" (6.28). Would be the "eigenfunctions" (6.28) of the operator $\mathcal{L}$ "true" $L^2(dt)$-functions, then the two-dimensional subspace generated by them will be invariant with respect to each of the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$. This means that for some matrix

$$F(\mu) = \begin{bmatrix} f_{++}(\mu) & f_{-+}(\mu) \\ f_{+-}(\mu) & f_{--}(\mu) \end{bmatrix},$$

which is constants with respect to t, the equality holds

$$\big(\mathcal{F}_E\psi_+(\,.\,,\mu)\big)(t) = f_{++}\psi_+(t,\mu) + f_{-+}\psi_-(t,\mu),$$
$$\big(\mathcal{F}_E\psi_-(\,.\,,\mu)\big)(t) = f_{+-}\psi_+(t,\mu) + f_{--}\psi_-(t,\mu)\,.$$

The matrix form of these equalities is:

$$\big(\mathcal{F}_E\psi(\,.\,,\mu)\big)(t) = F(\mu)\psi(t\,,\mu)\,. \tag{6.50a}$$

We show that

$$\big(\mathcal{F}^*_E\psi(\,.\,,\mu)\big)(t) = F^*(\mu)\psi(t\,,\mu)\,, \tag{6.50b}$$

where $F^*(\mu)$ is the matrix Hermitian conjugated to the matrix $F(\mu)$. However the functions $\psi_\pm(t,\mu)$ does not belong to $L^2(dt)$. So the operators $\mathcal{F}_E$, $\mathcal{F}^*_E$, considered as an operator acting in $L^2(dt)$, are not applicable to the functions $\psi_+(t,\mu), \psi_-(t,\mu)$. Nevertheless we can consider the Fourier integrals $\mathcal{F}_E\psi_\pm(t,\mu)$, $\mathcal{F}^*_E\psi_\pm(t,\mu)$ in some *Pickwick sense*. Namely we interpret the expressions $(\mathcal{F}_E\psi_\pm(\,.\,,\mu))(t)$ and $(\mathcal{F}^*_E\psi_\pm(\,.\,,\mu))(t)$ as

$$\big(\mathcal{F}_E\psi_\pm(\,.\,,\mu)\big)(t) = \lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \frac{1}{\sqrt{2\pi}} \int\limits_{\varepsilon}^{R} \xi^{-1/2\pm i\mu} e^{i\xi t}\, d\xi\,, \tag{6.51a}$$

$$\big(\mathcal{F}^*_E\psi_\pm(\,.\,,\mu)\big)(t) = \lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \frac{1}{\sqrt{2\pi}} \int\limits_{\varepsilon}^{R} \xi^{-1/2\pm i\mu} e^{-i\xi t}\, d\xi\,, \tag{6.51b}$$

In (6.51), $t \in (0,\infty)$, $\mu \in (0,\infty)$. It turns out that the limits in (6.51) exist and are uniform if t belongs to any fixed interval separated from zero and infinity. (We shall see this when calculating the integrals.) Changing variable in (6.51a): $\xi \to \xi/t$, and using the homogeneity properties of the functions $\psi_\pm(t,\mu)$ with respect to t, we obtain that

$$\big(\mathcal{F}_E\psi_+(\,.\,,\mu)\big)(t) = f_{-+}(\mu)\psi_-(t,\mu)\,, \tag{6.52a}$$
$$\big(\mathcal{F}_E\psi_-(\,.\,,\mu)\big)(t) = f_{+-}(\mu)\psi_+(t,\mu), \tag{6.52b}$$

where $t \in (0,\infty)$, $\mu > 0$, and

$$f_{-+}(\mu) = \lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \frac{1}{\sqrt{2\pi}} \int\limits_{\varepsilon}^{R} \xi^{-1/2+i\mu} e^{i\xi}\, d\xi\,, \tag{6.53a}$$

$$f_{+-}(\mu) = \lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \frac{1}{\sqrt{2\pi}} \int\limits_{\varepsilon}^{R} \xi^{-1/2-i\mu} e^{i\xi}\, d\xi\,. \tag{6.53b}$$

Changing variable in (6.51b): $\xi \to \xi/t$, and using the homogeneity properties of the functions $\psi_\pm(t,\mu)$ with respect to t, we obtain that

$$\big(\mathcal{F}_E^*\psi_+(\,.\,,\mu)\big)(t) = \overline{f_{+-}(\mu)}\psi_-(t,\mu)\,, \tag{6.54a}$$

$$\big(\mathcal{F}_E^*\psi_-(\,.\,,\mu)\big)(t) = \overline{f_{-+}(\mu)}\psi_+(t,\mu), \tag{6.54b}$$

where $t \in (0,\infty)$, $\mu > 0$. Let as calculate the integrals in (6.53). These integrals can be presented as

$$\lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \int\limits_{\varepsilon}^{R} \xi^{-1/2\pm i\mu} e^{i\xi}\, d\xi = e^{i\frac{\pi}{4}\mp\frac{\mu\pi}{2}} \lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \int\limits_{[-i\varepsilon,iR]} f(\zeta) d\zeta, \tag{6.55}$$

where

$$f(\zeta) = \zeta^{-1/2\pm i\mu} e^{-\zeta}\,, \quad \arg\zeta > 0 \;\text{ for }\; \zeta \in (0,\infty)\,. \tag{6.56}$$

Then we 'rotate' the ray of integration from the ray $(0,-i\infty)$ to the ray $(0,\infty)$. The function $f(\zeta)$ is holomorphic in the domain $\mathbb{C}\setminus(-\infty,0]$. According to Cauchy integral theorem,

$$\int\limits_{[-i\varepsilon,iR]} f(\zeta)\, d\zeta = \int\limits_{[\varepsilon,R]} f(\zeta)\, d\zeta + \int\limits_{\gamma_\varepsilon} f(\zeta) d\zeta + \int\limits_{\gamma_R} f(\zeta)\, d\zeta,$$

where γ_ε and γ_R are the arcs $-\pi/2 \le \arg z \le 0$, $|z| = \varepsilon$ and $|z|$ respectively. The functions $f(\zeta)$ grows as $|\varepsilon|^{-1/2}$ as $\zeta \in \gamma_\varepsilon$, $\varepsilon \to 0$, and the length of the arc γ_ε decays as ε, as $\varepsilon \to 0$. Therefore, $\int\limits_{\gamma_\varepsilon} f(\zeta)d\zeta \to 0$ as $\varepsilon \to 0$. Applying Jordan lemma to the function $f(\zeta)$ in the quadrant $-\pi/2 \le \arg\zeta \le 0$, we conclude that $\int\limits_{\gamma_R} f(\zeta)d\zeta \to 0$ as $R \to \infty$. Therefore

$$\lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \int\limits_{\varepsilon}^{R} \xi^{-1/2\pm i\mu} e^{i\xi}\, d\xi = e^{i\frac{\pi}{4}\mp\frac{\mu\pi}{2}} \int\limits_{0}^{+\infty} \xi^{-1/2\pm i\mu} e^{-\xi}\, d\xi\,.$$

The integral in the right hand side of the last formula is the Euler integral representing the Γ-function. Thus

$$\lim_{\substack{\varepsilon\to+0\\ R\to+\infty}} \int\limits_{\varepsilon}^{R} \xi^{-1/2\pm i\mu} e^{i\xi}\, d\xi = e^{i\frac{\pi}{4}\mp\frac{\mu\pi}{2}} \Gamma(1/2 \pm i\mu)\,, \quad -\infty < \mu < \infty\,, \tag{6.57}$$

and

$$f_{+-}(\mu) = \frac{1}{\sqrt{2\pi}}\, e^{i\frac{\pi}{4}+\frac{\mu\pi}{2}}\Gamma(1/2 - i\mu)\,, \tag{6.58a}$$

$$f_{-+}(\mu) = \frac{1}{\sqrt{2\pi}}\, e^{i\frac{\pi}{4}-\frac{\mu\pi}{2}}\Gamma(1/2 + i\mu)\,, \tag{6.58b}$$

where $\mu = \mu(\lambda)$ is defined by (6.32). Thus, the matrix $F = \begin{bmatrix} f_{++} & f_{-+} \\ f_{+-} & f_{--} \end{bmatrix}$ in (6.50) is of the form

$$F(\mu) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2\pi}}\, e^{i\frac{\pi}{4}-\frac{\mu\pi}{2}}\Gamma(1/2+i\mu) \\ \frac{1}{\sqrt{2\pi}}\, e^{i\frac{\pi}{4}+\frac{\mu\pi}{2}}\Gamma(1/2-i\mu) & 0 \end{bmatrix} \tag{6.59}$$

Thus the equalities (6.50) hold with the matrix $F(\mu)$ of the form (6.59). Given $x(t) \in L^2(dt)$, we apply the operators $\mathcal{F}_E$, $\mathcal{F}_E^*$ to the spectral expansion (6.37), (6.39). Applying the operators $\mathcal{F}_E$, $\mathcal{F}_E^*$ to the linear combination $\hat{x}(\mu)\psi(t,\mu)$, we should take into account that these operators act on functions of variable t and the coefficients $\hat{x}(\mu)$ of this linear combination do not depend on t. Therefore

$$\mathcal{F}_E\big(\hat{x}(\mu)\psi(\,.\,,\mu)\big)(t) = \hat{x}(\mu)\big(\mathcal{F}_E\psi(\,.\,,\mu)\big)(t)\,, \tag{6.60a}$$

$$\mathcal{F}_E^*\big(\hat{x}(\mu)\psi(\,.\,,\mu)\big)(t) = \hat{x}(\mu)\big(\mathcal{F}_E^*\psi(\,.\,,\mu)\big)(t)\,. \tag{6.60b}$$

Carry the operator $\mathcal{F}_E$ trough the integral in (6.39) and using (6.60), we obtain

$$(\mathcal{F}_E x)(t) = \int\limits_{\mu\in(0,\infty)} u_{\mathcal{F}_E}(\mu)\,\psi(t,\mu)\,\frac{d\mu}{2\pi}, \quad (\mathcal{F}_E^* x)(t) = \int\limits_{\mu\in(0,\infty)} u_{\mathcal{F}_E^*}(\mu)\,\psi(t,\mu)\,\frac{d\mu}{2\pi}, \tag{6.61a}$$

where

$$u_{\mathcal{F}_E}(\mu) = \hat{x}(\mu)F(\mu)\,, \quad u_{\mathcal{F}_E^*}(\mu) = \hat{x}(\mu)F^*(\mu)\,. \tag{6.61b}$$

Let us go to prove rigorously the formulas (6.61) expressing the spectral resolution of the vectors $\mathcal{F}_E\, x$ in terms of the spectral resolution (6.37) of the vector x. In this proof we use the following expressions for the absolute values of the entries of the matrix $F(\mu)$:

$$|f_{+-}(\mu)| = (1+e^{-2\pi\mu})^{-1/2}, \quad |f_{-+}(\mu)| = (1+e^{2\pi\mu})^{-1/2}, \quad 0 \le \mu < \infty\,. \tag{6.62}$$

The expressions (6.62) are derived from (6.58). Since

$$\Gamma(1/2+i\mu)\Gamma(1/2-i\mu)=\frac{\pi}{\cosh\pi\mu} \tag{6.63}$$

and the numbers $\Gamma(1/2\pm i\mu)$ are complex conjugated, then

$$|\Gamma(1/2\pm i\mu)|^2=\frac{2\pi}{e^{\pi\mu}+e^{-\pi\mu}},\quad 0\leq\mu<\infty\,. \tag{6.64}$$

The equalities (6.62) follows from the last formula and from (6.58). We remark that in particular

$$1/\sqrt{2}<|f_{+-}(\mu)|<1,\quad |f_{-+}(\mu)|<1/\sqrt{2},\ \ 0<\mu<\infty\,. \tag{6.65}$$

If μ runs over the interval $[0,\infty)$, then $|f_{+-}(\mu)|$ increases from $2^{-1/2}$ to 1 and $|f_{-+}(\mu)|$ decreases from $2^{-1/2}$ to 0. In particular,

$$\sup_{\mu\in(0,\infty)}|f_{+-}(\mu)|=\operatorname*{ess\,sup}_{\mu\in(0,\infty)}|f_{+-}(\mu)|=1\,. \tag{6.66}$$

From (6.58) and (6.64) it follows that

$$|f_{+-}(\mu)|^2+|f_{-+}(\mu)|^2=1\,, \tag{6.67}$$

$$|f_{+-}(\mu)|+|f_{-+}(\mu)|=\sqrt{1+\frac{1}{\cosh\,\pi\mu}}, \tag{6.68}$$

thus

$$1\leq|f_{+-}(\mu)|+|f_{-+}(\mu)|\leq\sqrt{2},\quad 0\leq\mu<\infty\,. \tag{6.69}$$

In view of the diagonal structure (6.59) of the matrix $F(\mu)$ and the estimates (6.65), (6.66) for its entries, the equalities

$$\|F(\mu)\|<1\quad\forall\,\mu\in(0,\infty) \tag{6.70a}$$

and

$$\operatorname*{ess\,sup}_{\mu\in(0,\infty)}\|F(\mu)\|=1\,. \tag{6.70b}$$

hold.

Theorem 6.3. *Let $x(t)\in L^2((0,\infty),dt)$, and $\hat{x}(\mu)$ be the Fourier transform of x,* (6.37)*:*

$$\hat{x}(\mu)=\int\limits_{(0,\infty)}x(t)\psi^*(t,\mu)\,dt\,,\quad \mu\in(0,\infty)\,.$$

Then the Fourier transforms $u_{\mathcal{F}_E}(\mu)$, $u_{\mathcal{F}_E^*}(\mu)$ *of the functions* $(\mathcal{F}_E\, x)(t)$, $(\mathcal{F}_E^*\, x)(t)$:

$$u_{\mathcal{F}_E}(\mu) = \int\limits_{\xi\in(0,\infty)} (\mathcal{F}_E\, x)(\xi)\psi^*(\xi,\mu)\, d\xi, \quad u_{\mathcal{F}_E^*}(\mu) = \int\limits_{\xi\in(0,\infty)} (\mathcal{F}_E^*\, x)(\xi)\psi^*(\xi,\mu)\, d\xi \tag{6.71}$$

are expressed in terms of $\hat{x}(\mu)$ *by the formula* (6.61b).

The functions $(\mathcal{F}_E\, x)(t)$, $(\mathcal{F}_E^*\, x)(t)$ *are expressed by the formula* (6.61a)*:*

$$(\mathcal{F}_E\, x)(t) = \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)F(\mu)\psi(t,\mu)\,\frac{d\mu}{2\pi},$$

$$(\mathcal{F}_E^*\, x)(t) = \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)F^*(\mu)\psi(t,\mu)\,\frac{d\mu}{2\pi}\,.$$

Proof. We substitute the expression

$$x(\xi) = \int\limits_{(0,\infty)} \hat{x}(\mu)\,\psi(\xi,\mu)\,\frac{d\mu}{2\pi}$$

for the function x, (6.39), into the formulas (6.1) and (6.2) which defines the truncated Fourier operator $\mathcal{F}_E$ and the adjoint operator $\mathcal{F}_E^*$. To curry the operators $\mathcal{F}_E$, $\mathcal{F}_E^*$ through the integral in (6.39), we have to change the order of integration in the iterated integrals

$$(\mathcal{F}_E x)(t) = \int\limits_0^\infty \bigg(\int\limits_0^\infty \hat{x}(\mu)\,\psi(\xi,\mu)\bigg)\,\frac{d\mu}{2\pi}\,e^{it\xi}\,d\xi\,, \tag{6.72a}$$

$$(\mathcal{F}_E^* x)(t) = \int\limits_0^\infty \bigg(\int\limits_0^\infty \hat{x}(\mu)\,\psi(\xi,\mu)\bigg)\,\frac{d\mu}{2\pi}\,e^{-it\xi}\,d\xi\,. \tag{6.72b}$$

Usual tool to justify the change of the order of integration is the Fubini theorem. However the Fubini theorem is not applicable to the iterated integrals (6.72). The function under the integral is not summable with respect to ξ.

To curry the operators $\mathcal{F}_E$, $\mathcal{F}_E^*$ through the integral in (6.39), we use a regularization procedure. Given $\varepsilon > 0$, we define *the regularization operator* $\mathcal{R}_\varepsilon :\ L^2((0,\infty)) \to L^2((0,\infty))$,

$$\mathcal{R}_\varepsilon x(t) = e^{-\varepsilon t}x(t)\,, \quad \forall x \in L^2((0,\infty))\,. \tag{6.73}$$

It is clear that for every $x \in L^2((0,\infty))$,

$$\|\mathcal{R}_\varepsilon x - x\|_{L^2((0,\infty))} \to 0 \quad \text{as} \quad \varepsilon \to +0\,.$$

The kernel of the operator $\mathcal{F}_E\mathcal{R}_\varepsilon$ can be calculated without difficulties. Let $x \in L^2((0,\infty)) \cap L^1((0,\infty))$. Then

$$(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{(0,\infty)} e^{i\xi s} e^{-\varepsilon\xi} x(\xi)\, d\xi\,,$$
$$(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{(0,\infty)} e^{-i\xi s} e^{-\varepsilon\xi} x(\xi)\, d\xi\,,$$

Substituting the expression (6.39) for the function $x(\xi)$ into the last formula, we present the functions $(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t)$, $(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t)$ as the iterated integrals

$$(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{(0,\infty)} e^{i\xi(t+i\varepsilon)} \Bigg(\int\limits_{(0,\infty)} \hat{x}(\mu)\, \psi(\xi,\mu)\, \frac{d\mu}{2\pi} \Bigg)\, d\xi\,, \quad \text{(6.74a)}$$

$$(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{(0,\infty)} e^{-i\xi(t-i\varepsilon)} \Bigg(\int\limits_{(0,\infty)} \hat{x}(\mu)\, \psi(\xi,\mu)\, \frac{d\mu}{2\pi} \Bigg)\, d\xi\,, \quad \text{(6.74b)}$$

We assume firstly that the function $\hat{x}(\mu)$ belongs to $L^2(d\mu) \cap L^1(d\mu)$. The Fubini theorem is applicable to each of the iterated integral (6.74). So for every fixed $\varepsilon > 0$ we can change the order of integration there. Changing the order, we obtain

$$(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu) \big(\mathcal{F}_E\mathcal{R}_\varepsilon \psi(\,.\,,\mu)\big)(t)\, d\mu\,, \qquad \text{(6.75a)}$$

$$(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu) \big(\mathcal{F}_E^*\mathcal{R}_\varepsilon \psi(\,.\,,\mu)\big)(t)\, d\mu\,, \qquad \text{(6.75b)}$$

where

$$\big(\mathcal{F}_E\mathcal{R}_\varepsilon \psi_\pm(\,.\,,\mu)\big)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \xi^{-1/2 \pm i\mu} e^{i(t+i\varepsilon)\xi}\, d\xi\,, \qquad \text{(6.76a)}$$

$$\big(\mathcal{F}_E^*\mathcal{R}_\varepsilon \psi_\pm(\,.\,,\mu)\big)(t) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \xi^{-1/2 \pm i\mu} e^{-i(t-i\varepsilon)\xi}\, d\xi\,. \qquad \text{(6.76b)}$$

The integrals in (6.76) can be calculated explicitly:

$$\big(\mathcal{F}_E\mathcal{R}_\varepsilon\psi_+(\,.\,,\mu)\big)(t) = f_{-+}(\mu)\psi_-(t+i\varepsilon,\,\mu), \tag{6.77a}$$
$$\big(\mathcal{F}_E\mathcal{R}_\varepsilon\psi_-(\,.\,,\mu)\big)(t) = f_{+-}(\mu)\psi_+(t+i\varepsilon,\,\mu)\,, \tag{6.77b}$$
$$\big(\mathcal{F}^*_E\mathcal{R}_\varepsilon\psi_+(\,.\,,\mu)\big)(t) = \overline{f_{+-}(\mu)}\psi_-(t-i\varepsilon,\,\mu), \tag{6.77c}$$
$$\big(\mathcal{F}^*_E\mathcal{R}_\varepsilon\psi_-(\,.\,,\mu)\big)(t) = \overline{f_{-+}(\mu)}\psi_+(t-i\varepsilon,\,\mu)\,, \tag{6.77d}$$

where $f_{+-}(\mu)$ and $f_{-+}(\mu)$ are the same that in (6.58), and

$$\psi_+(t\pm i\varepsilon,\mu) = (t\pm i\varepsilon)^{-1/2+i\mu} = e^{(-1/2+i\mu)(\ln|t+i\varepsilon|\pm i\arg(t+i\varepsilon))}\,, \tag{6.78a}$$
$$\psi_-(t\pm i\varepsilon,\mu) = (t\pm i\varepsilon)^{-1/2-i\mu} = e^{(-1/2-i\mu)(\ln|t+i\varepsilon|\pm i\arg(t+i\varepsilon))}\,. \tag{6.78b}$$

Here

$$0<\arg(t+i\varepsilon)<\pi/2 \quad \text{for} \;\; t>0,\,\varepsilon>0\,. \tag{6.79}$$

Formulas (6.78) are derived similarly to formulas (6.52). We change variable: $\xi\to\xi/|t+i\varepsilon|$, and then rotate the ray of integration. From (6.78) and (6.79) it follows that for every $t\in(0,\infty)$, $\mu\in(0,\infty)$

$$|\psi_+(t+i\varepsilon,\mu)|\le t^{-1/2},\;\; |\psi_-(t+i\varepsilon,\mu)|\le t^{-1/2}e^{\mu\pi/2}, \tag{6.80a}$$
$$|\psi_+(t-i\varepsilon,\mu)|\le t^{-1/2}e^{\mu\pi/2},\;\; |\psi_-(t-i\varepsilon,\mu)|\le t^{-1/2}\,. \tag{6.80b}$$

Taking into account the estimates (6.62), we obtain from (6.77) and (6.80) the estimates

$$\big|\big(\mathcal{F}_E\mathcal{R}_\varepsilon\psi_\pm(\,.\,,\mu)\big)(t)\big|\le t^{-1/2}, \tag{6.81a}$$
$$\big|\big(\mathcal{F}^*_E\mathcal{R}_\varepsilon\psi_\pm(\,.\,,\mu)\big)(t)\big|\le t^{-1/2}\,, \tag{6.81b}$$

which hold for every $\mu>0$, $t>0$ and $\varepsilon>0$. In particular, the expressions in the right hand sides of (6.81) do not depend on ε. Moreover, from (6.77) it follows that for every fixed $\mu>0$ and $t>0$, there exist the limits

$$\lim_{\varepsilon\to+0}\big(\mathcal{F}_E\mathcal{R}_\varepsilon\psi(\,.\,,\mu)\big)(t) = F(\mu)\psi(t,\mu)\,, \tag{6.82a}$$
$$\lim_{\varepsilon\to+0}\big(\mathcal{F}^*_E\mathcal{R}_\varepsilon\psi(\,.\,,\mu)\big)(t) = F^*(\mu)\psi(t,\mu)\,. \tag{6.82b}$$

Using the Lebesgue dominating convergence theorem, we conclude that for every fixed $t > 0$

$$\lim_{\varepsilon\to+0} \int\limits_{(0,\infty)} \hat{x}(\mu)\big(\mathcal{F}_E\mathcal{R}_\varepsilon\psi(\,.\,,\mu)\big)(t)\,d\mu = \int\limits_{(0,\infty)} \hat{x}(\mu)\big(F(\mu)\psi(t,\mu)\big)\,d\mu\,,$$

$$\lim_{\varepsilon\to+0} \int\limits_{(0,\infty)} \hat{x}(\mu)\big(\mathcal{F}_E^*\mathcal{R}_\varepsilon\psi(\,.\,,\mu)\big)(t)\,d\mu = \int\limits_{(0,\infty)} \hat{x}(\mu)\big(F^*(\mu)\psi(t,\mu)\big)\,d\mu\,.$$

Involving (6.75a), we see that

$$\lim_{\varepsilon\to+0}(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu)\big(F(\mu)\psi(t,\mu)\big)\,d\mu\,,$$

$$\lim_{\varepsilon\to+0}(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu)\big(F^*(\mu)\psi(t,\mu)\big)\,d\mu\,.$$

for every fixed $t > 0$. From the other hand,

$$\lim_{\varepsilon\to+0}\|(\mathcal{F}_E\mathcal{R}_\varepsilon x)(t) - (\mathcal{F}_E x)(t)\|_{L^2(dt)} = 0\,,$$

$$\lim_{\varepsilon\to+0}\|(\mathcal{F}_E^*\mathcal{R}_\varepsilon x)(t) - (\mathcal{F}_E^* x)(t)\|_{L^2(dt)} = 0\,,$$

Comparing the last formulas, we obtain that

$$(\mathcal{F}_E x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu)F(\mu)\psi(t,\mu)\,d\mu\,, \tag{6.83a}$$

$$(\mathcal{F}_E^* x)(t) = \int\limits_{(0,\infty)} \hat{x}(\mu)F^*(\mu)\psi(t,\mu)\,d\mu\,, \tag{6.83b}$$

for every $x(t) \in L^2(dt)$ for which $\hat{x}(\mu) \in L^2(d\mu)\cap L^1(d\mu)$. Since the set $L^2(d\mu)\cap L^1(d\mu)$ is dense in $L^2(d\mu)$, the last equality can be extended to all $x(t) \in L^2(dt)$. To justify such extension, one should involve the Parseval equality taking into account that the matrix $F(\mu)$ is bounded, (6.62). □

4. To find the spectrum and the resolvent of the operator $\mathcal{F}_E$, we need to develop a functional calculus related to the 'orthogonal' system $\psi(t,\mu)$ of eigenfunctions of the operator $\mathcal{L}$.

Definition 6.5. *Let $M(\mu)$,*

$$M(\mu) = \begin{bmatrix} m_{++}(\mu) & m_{-+}(\mu) \\ m_{+-}(\mu) & m_{--}(\mu) \end{bmatrix}, \qquad 0 < \mu < \infty\,, \tag{6.84}$$

be a measurable 2×2 matrix function which is defined almost everywhere on the half-axis $0 < \mu < \infty$.

Assume that the function $M(\mu)$ is essentially bounded:

$$\operatorname*{ess\,sup}_{\mu \in (0,\infty)} \|M(\mu)\| < \infty\,, \tag{6.85}$$

where for each μ, the expression $\|M(\mu)\|$ means the norm of the matrix $M(\mu)$ considered as an operator in the two-dimensional complex Euclidean space, and $\operatorname{ess\,sup}$ is the so-called essential supremum of a real-valued measurable function.
The operator $M(\mathcal{L})$,

$$M(\mathcal{L}): \ L^2((0,\infty), dt) \to L^2((0,\infty), dt)\,, \tag{6.86}$$

is defined as follows. Given $x(t) \in L^2((0,\infty),\, dt)$, then

$$(M(\mathcal{L})x)(t) \stackrel{\text{def}}{=} \int\limits_{\mu \in (0,\infty)} \hat{x}(\mu) M(\mu) \psi(t,\mu)\, \frac{d\mu}{2\pi}\,, \tag{6.87a}$$

where $\hat{x}(\mu)$ is expressed from $x(t)$ by

$$\hat{x}(\mu) = \int\limits_{(0,\infty)} x(t) \psi^*(t,\mu)\, dt\,, \quad \mu \in (0,\infty)\,. \tag{6.87b}$$

Remark 6.2. *The operator $M(\mathcal{L})$ can be considered as an integral operator:*

$$x(t) \to \int\limits_{(0,\infty)} x(\xi) K_{M(\mathcal{L})}(\xi, t)\, d\xi$$

with the kernel

$$K_{M(\mathcal{L})}(\xi, t) = \int\limits_{\mu \in (0,\infty)} \psi^*(\xi,\, \mu) M(\mu) \psi(t,\mu)\,.$$

However the kernel $K_{M(\mathcal{L})}(\xi, t)$ may be a distribution.

Remark 6.3. *Let us motivate the notation $M(\mathcal{L})$ which appears in* (6.86)*. The representation* (6.87) *can be considered as the expansion of the vector x related to the resolution of identity generated by the operator $\mathcal{L}$. The operator $M(\mathcal{L})$ can be considered as a* function M of the operator $\mathcal{L}$*, where, in contrast with the traditional functional calculus for operators, M is a* matrix-valued*, rather than a scalar-valued, function. Such a definition is consistent because the multiplicity of the spectrum of $\mathcal{L}$ at the point λ coincides with the dimension of the matrix $M(\mu(\lambda))$.*

Remark 6.4. Theorem 6.3 *means that the truncated operator $\mathcal{F}_E$ and its adjoint $\mathcal{F}_E^*$ can be considered as the function $F(\mathcal{L})$ and $F^*(\mathcal{L})$ of the operator $\mathcal{L}$ respectively*:

$$\mathcal{F}_E = F(\mathcal{L}), \quad \mathcal{F}_E^* = F^*(\mathcal{L})\,, \tag{6.88}$$

where the 2×2*-matrix-function $F(\mu)$ is defined in* (6.59)*.*

Remark 6.5. In more details, the function $(M(\mathcal{L})x)(t)$ is defined as follows. Since $x(t) \in L^2(0,\infty), dt)$, the vector-row function $\hat{x}(\mu)$,

$$\hat{x}(\mu) = \int\limits_{\xi\in(0,\infty)} x(\xi)\,\psi^*(\xi,\mu)\,d\xi\,,$$

is a well defined $L^2((0,\infty), d\mu)$-function, and

$$\int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\,\frac{d\mu}{2\pi} = \int\limits_{t\in(0,\infty)} |x(t)|^2 dt\,.$$

The vector-row function

$$\hat{m}(\mu) = \hat{x}(\mu)M(\mu)$$

also is a $L^2((0,\infty), d\mu)$-function:

$$\int\limits_{\mu\in(0,\infty)} \hat{m}(\mu)\hat{m}^*(\mu)\,d\mu \leq \Big(\operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M(\mu)\|\Big)^2 \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\,d\mu.$$

Then the integral in the right hand side of the expression

$$m(t) = \int\limits_{\mu\in(0,\infty)} \hat{m}(\mu)\psi(t,\mu)\,\frac{d\mu}{2\pi}\,, \tag{6.89}$$

determines a function $m(t)$, which belongs to $L^2((0,\infty),\,dt)$:

$$\int\limits_{t\in(0,\infty)} |m(t)|^2 dt = \int\limits_{\mu\in(0,\infty)} \hat{m}(\mu)\hat{m}^*(\mu)\,\frac{d\mu}{2\pi}\,.$$

(See Remark (6.1)). We set

$$(M(\mathcal{L})x)(t) = m(t)\,.$$

From the above chain of equalities and inequalities it follows that

$$\int\limits_{t\in(0,\infty)} \big|(M(\mathcal{L})x)(t)\big|^2 dt \leq \Big(\operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M(\mu)\Big)^2 \int\limits_{t\in(0,\infty)} |x(t)|^2 dt\,. \quad (6.90)$$

Theorem 6.4. *Under the condition* (6.85)*, the operator* $M(\mathcal{L})$ *is a bounded operator in the space* $L^2((0,\infty),\,dt)$*, and*

$$\|M(\mathcal{L})\| = \operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M(\mu)\|\,. \quad (6.91)$$

Proof. The estimate

$$\|M(\mathcal{L})\| \leq \operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M(\mu)\|$$

of the norm $\|M(\mathcal{L})\|$ from above is the inequality (6.90).

Proving the inverse inequality, we assume that $\operatorname*{ess\,sup}\limits_{\mu\in(0,\infty)} \|M(\mu)\| > 0$. We take arbitrary ε, $0<\varepsilon<1$ and fix it. According to the definition of the notion of essential supremum, there exists the set S, $S\subseteq(0,\infty)$, such that

$$\operatorname{mes} S > 0 \;\text{ and }\; \|M(\mu)\| \geq (1-\varepsilon)\operatorname*{ess\,sup}_{\nu\in(0,\infty)} \|M(\nu)\| \;\;\forall\,\mu\in S\,.$$

For every $\mu\in S$, take $\hat{x}(\mu) = \begin{bmatrix}\hat{x}_1(\mu) & \hat{x}_2(\mu)\end{bmatrix}$, $\hat{x}(\mu)\neq 0$, such that on this vector the norm of the matrix $M(\mu)$ is almost attained. This means that if

$$\hat{m}(\mu) = M(\mu)\hat{x}(\mu), \quad \hat{m}(\mu) = \begin{bmatrix}\hat{m}_1(\mu) & \hat{u}_2(\mu)\end{bmatrix}\,,$$

then

$$\hat{m}(\mu)\hat{m}^*(\mu) \geq (1-\varepsilon)^2\|M(\mu)\|^2\hat{x}(\mu)\hat{x}^*(\mu)\,.$$

Thus, for $\mu \in S$ the inequality

$$\hat{m}(\mu)\hat{m}^*(\mu) \geq \big(1-\varepsilon\big)^4 \big(\operatorname*{ess\,sup}_{\nu\in(0,\infty)} \|M(\nu)\|\big)^2\, \hat{x}(\mu)\hat{x}^*(\mu) \tag{6.92}$$

holds. For $\mu \notin S$, we set $\hat{x}(\mu) = 0$. Then $\hat{m}(\mu) = 0$ for $\mu \notin S$ and the inequality (6.92) trivially holds for such μ. Thus, $\hat{x}(\mu) \not\equiv 0$ and the inequality (6.92) holds for *every* $\mu > 0$. If the functions $x(t)$ and $m(t)$ are defined from $\hat{x}(\mu)$ and $\hat{m}(\mu)$ according to (6.39) and (6.89) respectively, then

$$\int\limits_{t\in(0,\infty)} |m(t)|^2\, dt = \int\limits_{\mu\in(0,\infty)} \hat{m}(\mu)\hat{m}^*(\mu)\, \frac{d\mu}{2\pi} \geq$$

$$\geq (1-\varepsilon)^4 \big(\operatorname*{ess\,sup}_{\nu\in(0,\infty)} \|M(\nu)\|\big)^2 \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu) \frac{d\mu}{2\pi} =$$

$$=\geq (1-\varepsilon)^4 \big(\operatorname*{ess\,sup}_{\nu\in(0,\infty)} \|M(\nu)\|\big)^2 \int\limits_{t\in(0,\infty)} |x(t)|^2\, dt\,.$$

Thus, for every $\varepsilon > 0$ there exists $x \neq 0$ such that

$$\|M(\mathcal{L})x\|^2 \geq (1-\varepsilon)^4 \big(\operatorname*{ess\,sup}_{\nu\in(0,\infty)} \|M(\nu)\|\big)^2 \|x\|^2\,.$$

Hence, the estimate of the norm

$$\|M(\mathcal{L}\| \geq \operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M(\mu)\|$$

of the norm $\|M(\mathcal{L})\|$ from below holds. □

Corollary 6.1. *As we already mentioned, the operator $\mathcal{F}_E$ can be considered as the function $F(\mathcal{L})$ of the operator $\mathcal{L}$.* (Remark 6.4). *Therefore, according to* Theorem 6.4 *and* (6.70b)*, the equality*

$$\|\mathcal{F}_E\| = 1 \tag{6.93}$$

holds.

Remark 6.6. *Despite the equality* (6.93)*, for every $x \in L^2(0,\infty)$, $\|x\|_{L^2(dt)} \neq 0$, the inequality*

$$\|\mathcal{F}_E x\|_{L^2(dt)} < \|x\|_{L^2(dt)} \tag{6.94}$$

holds.

The Parseval equality applied to the functions $x(t)$ and $\mathfrak{F}x(t)$ gives:

$$\|x\|^2_{L^2(dt)} = \int\limits_{(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\frac{d\mu}{2\pi}$$

and

$$\|\mathfrak{F}_E x\|^2_{L^2(dt)} = \int\limits_{(0,\infty)} \hat{x}(\mu)F(\mu)F^*(\mu)\hat{x}^*(\mu)\frac{d\mu}{2\pi}\,.$$

According to (6.70), $F(\mu)F^* < I$ for every $\mu \in (0,\infty)$. Thus,

$$\hat{x}(\mu)F(\mu)F^*(\mu)\hat{x}^*(\mu) < \hat{x}(\mu)\hat{x}^*(\mu)$$

for any $\mu \in (0,\infty)$ such that $\hat{x}(\mu)\hat{x}^*(\mu) > 0$. Therefore the inequality (6.94) holds if $x \neq 0$.

The inequality (6.94) can also be proved by another way. For $x(t) \in L^2((0,\infty))$, its Fourier transform (non-truncated) $(\mathfrak{F}x)(t)$ belongs to the Hardy class H^2_+. Therefore, the function $(\mathfrak{F}x)(t)$ can not vanish on the set of positive Lebesgue measure. In particular, $\int\limits_{(-\infty,0)} |(\mathfrak{F}x)(t)|^2 dt > 0\,.$ Therefore

$$\begin{aligned} &\|x\||^2_{L^2((0,\infty))} - \|(\mathfrak{F}_E x)\|^2_{L^2((0,\infty))} = \\ &\|\|(\mathfrak{F}x)\|^2_{L^2((-\infty,\infty))} - \|(\mathfrak{F}_E x)\|^2_{L^2((0,\infty))} = \int\limits_{(-\infty,0)} |(\mathfrak{F}x)(t)|^2 dt > 0\,. \end{aligned}$$

Theorem 6.5. *The mapping $M(\mu) \to M(\mathcal{L})$ which was introduced in* Definition 6.5 *is a homomorphism of the algebra of 2×2-matrix functions $M(\mu)$, which are defined on the half-axis $0 < \mu < \infty$ and bounded there, into the algebra of bounded linear operators acting in $L^2((0,\infty),\, dt)$:*

1. *If $M(\mu) \equiv I$ for almost every $\mu \in (0,\infty)$, then $M(\mathcal{L}) = \mathfrak{I}$, where I is the 2×2 identity matrix, and $\mathfrak{I}$ is the identity operator in $L^2((0,\infty),\, dt)$.*
2. *If $M(\mu) = \alpha_1 M_1(\mu) + \alpha_2 M_2(\mu)$, where $\alpha_1,\, \alpha_2 \in \mathbb{C}$, then $M(\mathcal{L}) = \alpha_1 M_1(\mathcal{L}) + \alpha_2 M_2(\mathcal{L})$.*
3. *If $M(\mu) = M_1(\mu)\cdot M_2(\mu)$, then $M(\mathcal{L}) = M_2(\mathcal{L})\cdot M_1(\mathcal{L})$.*
4. *$M^*(\mathcal{L}) = (M(\mathcal{L}))^*$.*

Proof. Statement 1 of Theorem 6.5 is a consequence of Definition 6.5 and of Theorem 6.2, Statement 2 (See the equality (6.39).) Statements 2 and 3 are direct consequences of Definition 6.5.

Let us prove Statement 4. Given a matrix-function $M(\mu)$ and $x(t),\ y(t) \in L^2(0,\infty)$, we have to check the equality

$$\langle M(\mathcal{L})x, y\rangle = \langle x, M^*(\mathcal{L})y\rangle \tag{6.95}$$

Let $\hat{x}$ and $\hat{y}$ be the "Fourier transforms" of x and y with respect to eigenfunctions $\psi(t,\mu)$ of the operator $\mathcal{L}$:

$$\hat{x}(\mu) = \int\limits_{t\in(0,\infty)} x(t)\psi^*(t,\mu)\,dt, \quad \hat{y}(\mu) = \int\limits_{t\in(0,\infty)} y(t)\psi^*(t,\mu)\,dt\,.$$

According to Definition 6.5, the Fourier transforms of the functions $M(\mathcal{L})x$ and $M^*(\mathcal{L})y$ are:

$$(\widehat{M(\mathcal{L})}x)(\mu) = M(\mu)\hat{x}(\mu), \quad (\widehat{M^*(\mathcal{L})}y)(\mu) = M^*(\mu)\hat{y}(\mu)\,.$$

According the Parseval identity, see (6.38), the equality (6.95) is equivalent to the equality

$$\int\limits_{\mu\in(0,\infty)} \big(\hat{x}(\mu)M(\mu)\big)\,\hat{y}^*(\mu)\,\frac{d\mu}{2\pi} = \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\,\big(\hat{y}(\mu)M^*(\mu)\big)^*\,\frac{d\mu}{2\pi}\,.$$

The last equality is evident. According to the rules of the matrix algebra, for every 1×2 vector rows $\hat{x}(\mu)$, $\hat{y}(\mu)$ and for every 2×2 matrix $M(\mu)$, the equality

$$\big(\hat{x}(\mu)M(\mu)\big)\,\hat{y}^*(\mu) = \hat{x}(\mu)\,\big(\hat{y}(\mu)M^*(\mu)\big)^*$$

hold. $\square$

Remark 6.7. Using the expression (6.59) for the matrix function $F(\mu)$ and the identity (6.63), we obtain

$$F(\mu)F^*(\mu) = \begin{bmatrix} \frac{1}{2}\,\frac{e^{-\pi\mu}}{\cosh\pi\mu} & 0 \\ 0 & \frac{1}{2}\,\frac{e^{\pi\mu}}{\cosh\pi\mu} \end{bmatrix},$$

$$F^*(\mu)F(\mu) = \begin{bmatrix} \frac{1}{2}\,\frac{e^{\pi\mu}}{\cosh\pi\mu} & 0 \\ 0 & \frac{1}{2}\,\frac{e^{-\pi\mu}}{\cosh\pi\mu} \end{bmatrix}.$$

and

$$F(\mu)F^*(\mu) + F^*(\mu)F(\mu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{6.96a}$$

$$F(\mu)F^*(\mu) - F^*(\mu)F(\mu) = \begin{bmatrix} -\tanh \pi\mu & 0 \\ 0 & \tanh \pi\mu \end{bmatrix}. \tag{6.96b}$$

From (6.96a) and (6.88) and Theorem 6.5 the equality

$$\mathcal{F}_E^*\mathcal{F}_E + \mathcal{F}_E\mathcal{F}_E^* = \mathcal{I} \tag{6.97}$$

follows, where $\mathcal{I}$ is the identity operator in $L^2(0,\infty)$.

Definition 6.6. *Let $M(\mu)$ be a measurable 2×2 matrix function which is defined on the half-axis $0 < \mu < \infty$ and is essentially bounded there, that the condition* (6.85) *is satisfied. The matrix function $M(\mu)$ is said to be invertible if the matrix $M(\mu)$ is invertible for almost every μ, and*

$$\operatorname*{ess\,sup}_{\mu\in(0,\infty)} \|M^{-1}(\mu)\| < \infty\,. \tag{6.98}$$

Theorem 6.6. *Let $M(\mu)$ be a measurable 2×2 matrix function,* (6.84), *which is defined almost everywhere on the half-axis $0 < \mu < \infty$ and essentially bounded there. Let $M(\mathcal{L})$ be the operator which is defined according to the functional calculus introduced in* Definition 6.5.

Then:

1. *The operator $M(\mathcal{L})$ is an invertible operator in $L^2((0,\infty), dt)$ if and only if the matrix function $M(\mu)$ is invertible in the sense of* Definition 6.6.
2. *The operator $M(\mathcal{L})$ is not invertible if and only if one of two cases, which do not exclude each other, takes place:*
   a). *There exists x such that $x \neq 0$, but $M(\mathcal{L})x = 0$;*
   b). *There exists a sequence $\{x_n\}_{1\leq n<\infty}$ such that $x_n \not\to 0$ but $M(\mathcal{L})x_n \to 0$ as $n \to \infty$.*
3. *The case* a). *takes place if and only if there exists a set S_{ni} of positive measure such that the matrix $M(\mu)$ is not invertible for any $\mu \in S_{\mathrm{ni}}$.*
   If there exists a set S_{nb} of positive measure such that the matrix $M(\mu)$ is invertible for any $\mu \in S_{\mathrm{nb}}$, but the inverse matrix function $M^{-1}(\mu)$ is not essentially bounded on S_{nb}: $\operatorname*{ess\,sup}_{\mu\in S_{\mathrm{nb}}} \|M^{-1}(\mu)\| = \infty$, then the case b). *takes place.*

Proof. If the matrix function $M(\mu)$ is invertible, then the operator $M^{-1}(\mathcal{L})$ is well defined. According to Theorem 6.5, the equalities $M(\mathcal{L})M^{-1}(\mathcal{L}) = \mathfrak{I}$ and $M^{-1}(\mathcal{L})M(\mathcal{L}) = \mathfrak{I}$ hold. Thus the operator $M(\mathcal{L})$ is invertible, and

$$(M(\mathcal{L}))^{-1} = M^{-1}(\mathcal{L})\,. \tag{6.99}$$

Let us assume now that the matrix function $M(\mu)$ is not invertible. Then either there exists the set S_{ni}, $\operatorname{mes} S_{\mathrm{ni}} > 0$, such that the matrix $M(\mu)$ is not invertible for every $\mu \in S_{\mathrm{ni}}$, or there exists the set S_{nb}, $\operatorname{mes} S_{\mathrm{nb}} > 0$, such that the matrix $M(\mu)$ is invertible for every $\mu \in S_{\mathrm{nb}}$, but $\operatorname*{ess\,sup}\limits_{\mu\in S_{\mathrm{nb}}} \|M^{-1}(\mu)\| = \infty$. (These two cases do not exclude each other.) Considering the first case, we choose and fix a set S_{ni} for which $\operatorname{mes} S_{\mathrm{ni}} > 0$. For every $\mu \in S_{\mathrm{ni}}$ there exists the vector-row $\hat{x}(\mu) = \big[\hat{x}_1(\mu), \hat{x}_2(\mu)\big]$ such that $\hat{x}(\mu) \neq 0$, but $\hat{x}(\mu)M(\mu) = 0$. Set $\hat{x}(\mu) = 0$ for $\mu \notin S_{\mathrm{ni}}$. Now the row function $\hat{x}(\mu)$ is defined almost everywhere on $(0, \infty)$. Since the matrix function $M(\mu)$ is measurable, the row function $\hat{x}(\mu)$ can be chosen measurable as well. We also normalize $\hat{x}(\mu)$ so that $\int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\frac{d\mu}{2\pi} = 1$. The $L^2((0,\infty),\, dt)$-function $x(t)$ is constructed from this $\hat{x}(\mu)$ according to (6.40). Then $\|x\| = 1$, but $M(\mathcal{L})x = 0$.

Inversely, assume that for some $x \in L^2(0,\infty)$, $\|x\| \neq 0$, the equality $M(\mathcal{L})x = 0$ holds. Set $S_{\mathrm{no}} = \{\mu : \hat{x}(\mu) \neq 0\}$. Since $\|x\| \neq 0$, $\operatorname{mes} S_{\mathrm{no}} > 0$. Since the row function $\hat{x}(\mu)M(\mu)$ vanishes almost everywhere on $(0, \infty)$, matrix $M(\mu)$ is not invertible for almost every $\mu \in S_{\mathrm{no}}$.

Considering the second case, we choose and fix a set S_{nb} for which $\operatorname{mes} S_{\mathrm{nb}} > 0$, the function $M^{-1}(\mu)$ is defined almost everywhere on the set S_{nb}, but is not essentially bonded there. Then for any $\varepsilon > 0$, there exists a measurable row function $\hat{u}(\mu)$ on S_{nb} such that $\|u(\mu)\| \leq \varepsilon\|u(\mu)M^{-1}(\mu)\|$ for all $\mu \in S_{\mathrm{nb}}$ and $\operatorname{mes}\{\mu : u(\mu) \neq 0\} > 0$. Denoting $\hat{x}(\mu) = \hat{u}(\mu)M^{-1}(\mu)$, we come to the function $\hat{x}(\mu)$ defined on the set S_{nb}, which is not zero identically: $\operatorname{mes}\{\mu : \hat{x}(\mu) \neq 0\} > 0$, and for which the inequality

$$\|\hat{x}(\mu)M(\mu)\| \leq \varepsilon\|\hat{x}(\mu)\|$$

holds almost everywhere on the set S_{nb}. We extend the function $\hat{x}(\mu)$ from the set S_{nb} on the whole half-axis $(0, \infty)$ setting $\hat{x}(\mu) = 0$ for $\mu \notin S_{\mathrm{nb}}$. For the extended function $\hat{x}(\mu)$, the above inequality holds almost everywhere on $(0, \infty)$. The function $\hat{x}(\mu)$ can be chosen to

be measurable. Without loss of generality we may assume that the function $\hat{x}(\mu)$ is normalized so that $\int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\frac{d\mu}{2\pi} = 1$. The $L^2((0,\infty),\,dt)$ - function $x(t)$ is constructed from this $\hat{x}(\mu)$ according to (6.39). For this function, $\|x\| = 1$, but $\|M(\mathcal{L})x\| \leq \varepsilon$. □

Theorem 6.7.

1. *The spectrum* $\sigma_{\mathfrak{F}_E}$ *of the truncated Fourier operator* $\mathfrak{F}_E$, $E = (0,\infty)$, *considered as an operator in* $L^2(0,\infty)$, *is the interval* $\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]$*:*
$$\sigma_{\mathfrak{F}_E} = \Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]. \tag{6.100}$$
The spectrum of the adjoint operator $\mathfrak{F}^*_E$, $E = (0,\infty)$ *is the interval* $\Big[-\frac{1}{\sqrt{2}}\,e^{-i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{-i\pi/4}\Big]$*:*
$$\sigma_{\mathfrak{F}^*_E} = \Big[-\frac{1}{\sqrt{2}}\,e^{-i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{-i\pi/4}\Big]. \tag{6.101}$$

2. *The operators* $\mathfrak{F}_E$, $\mathfrak{F}^*_E$ *have no eigenvalues: if for some* $z \in \mathbb{C}$, *either* $(\mathfrak{F}_E - \lambda\,\mathfrak{I})x = 0$ *or* $(\mathfrak{F}^*_E - \lambda\,\mathfrak{I})x = 0$, *where* $x \in L^2(0,\infty)$, *then* $x = 0$.

3. *If* $z \in \sigma_{\mathfrak{F}_E}$, *then there exists a sequence* $\{x_n\}_{1\leq n<\infty}$, $x_n \in L^2(0,\infty)$, $\|x_n\| = 1\,\forall\,n$, *but* $\|(\mathfrak{F}_E - \mathfrak{I})x_n\| \to 0$ *as* $n \to \infty$.

4. *If* $z \in \sigma_{\mathfrak{F}_E}$, *then the image of the operator* $z\mathfrak{I} - \mathfrak{F}$ *is a dense (non-closed) subspace of the space* $L^2((0,\infty))$.

5. *If* $z \notin \sigma_{\mathfrak{F}_E}$, *the the resolvent* $(z\mathfrak{I} - \mathfrak{F}_E)^{-1}$ *of the operator* $\mathfrak{F}_E$ *can be presented as a matrix-function of the operator* $\mathcal{L}$*:*
$$(z\mathfrak{I} - \mathfrak{F}_E)^{-1} = M(\mathcal{L}), \tag{6.102a}$$
where
$$M(\mu) = (zI - F(\mu))^{-1}\,. \tag{6.102b}$$

Lemma 6.3. *The norm of an arbitrary* 2×2 *matrix* $M = \left[\begin{smallmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{smallmatrix}\right]$ *can be estimated from its entries as follows:*
$$\tfrac{1}{2}\,\mathrm{trace}\,(M^*M) \leq \|M\|^2 \leq \mathrm{trace}\,(M^*M)\,. \tag{6.103}$$

Assuming that det $M \neq 0$, *the norm of the inverse matrix* M^{-1} *can be estimated as follows:*

$$(\det\, M)|^{-2}\,\mathrm{trace}\,(M^{*}M) - \frac{2}{\mathrm{trace}\,(M^{*}M)} \leq$$
$$\leq \|M^{-1}\|^{2} \;\leq\; |(\det\, M)|^{-2}\,\mathrm{trace}\,(M^{*}M) \qquad (6.104)$$

where

$$\mathrm{trace}\, M^{*}M = |m_{11}|^{2} + |m_{12}|^{2} + |m_{21}|^{2} + |m_{22}|^{2}\,. \qquad (6.105)$$

Proof. Let s_0 and s_1 be singular values of the matrix M, that is

$$0 < s_1 \leq s_0\,, \qquad (6.106)$$

and the numbers s_0^2, s_1^2 are eigenvalues of the matrix $M^{*}M$. Then

$$\|M\| = s_0, \quad \|M^{-1}\| = s_1^{-1},$$
$$\mathrm{trace}(M^{*}M) = s_0^2 + s_1^2\,, \quad |\det(M)|^2 = \det\,(M^{*}M) = s_0^2 \cdot s_1^2\,.$$

Therefore the inequality (6.103) takes the form

$$\frac{1}{2}(s_0^2 + s_1^2) \leq s_0^2 \leq (s_0^2 + s_1^2)\,,$$

and the inequality (6.104) takes the form

$$(s_0 s_1)^{-2}(s_0^2 + s_1^2) \;-\; \frac{2}{s_0^2 + s_1^2} \;\leq\; s_1^{-2} \;\leq\; (s_0 s_1)^{-2}(s_0^2 + s_1^2)\,.$$

The last inequalities hold for arbitrary numbers s_0, s_1 which satisfy the inequalities (6.106). □

Proof of Theorem 6.7. The proof is based on Theorem 6.6 and the interpretation of the operator $\mathcal{F}_E$ as a matrix function of the operator $\mathcal{L}$:

$$\mathcal{F}_E = F(\mathcal{L}).$$

(See Remark 6.4.) We apply Theorem 6.6 to the matrix function $M(\mu) = zI - F(\mu)$, where the 2×2-matrix-function $F(\mu)$ is defined in (6.59), and z is a complex number:

$$zI - F(\mu) = \begin{bmatrix} z & -f_{-+}(\mu) \\ -f_{+-}(\mu) & z \end{bmatrix}, \qquad (6.107)$$

where the expressions for $f_{+-}(\mu)$, $f_{-+}(\mu)$ are presented in (6.58). According to Theorem 6.6, the invertibility of the operator $z\mathcal{I} - \mathcal{F}_E$ is equivalent to the condition: *the matrix function $zI - F(\mu)$ is invertible for every $\mu \in [0, \infty)$, and the value in the right hand side of the following formula*

$$\left\|\left(z\mathcal{I} - \mathcal{F}_E\right)^{-1}\right\| = \sup_{\mu \in [0,\infty)} \left\|\left(zI - F(\mu)\right)^{-1}\right\| \tag{6.108}$$

is finite. If this value is finite, the formula (6.108) gives the expression for the norm of the operator $\left(z\mathcal{I} - \mathcal{F}_E\right)^{-1}$.

Let $D(z, \mu)$ be the determinant of the matrix $zI - F(\mu)$:

$$D(z, \mu) = \det(zI - F(\mu))\,. \tag{6.109}$$

The matrix $M(\mu) = zI - F(\mu)$ is invertible if and only if if its determinant $D(z, \mu)$ is different from zero. We apply Lemma 6.3 to this matrix function. According to (6.67) and (6.105),

$$\operatorname{trace}\left((zI - F(\mu))^*(zI - F(\mu))\right) = 2|z|^2 + 1\,. \tag{6.110}$$

Thus, the inequalities (6.104), applied to the matrix $M = zI - F(\mu))$, take the form

$$\begin{aligned} &|D(z, \mu)|^{-2}\left(2|z|^2 + 1\right) - \frac{2}{2|z|^2 + 1} \leq \\ &\qquad \leq \|(zI - F(\mu))^{-1}\|^2 \leq |D(z, \mu)|^{-2}\left(2|z|^2 + 1\right)\,. \end{aligned} \tag{6.111}$$

In particular,

$$\begin{aligned} &(2|z|^2 + 1)\left(\inf_{\mu \in (0,\infty)} |D(z, \mu)|\right)^{-2} - \frac{2}{2|z|^2 + 1} \leq \\ &\leq \sup_{\mu \in (0,\infty)} \|(zI - F(\mu))^{-1}\|^2 \leq \left(2|z|^2 + 1\right)\left(\inf_{\mu \in (0,\infty)} |D(z, \mu)|\right)^{-2}. \end{aligned} \tag{6.112}$$

From (6.107) and the identity (6.63) it follows that

$$D(z, \mu) = z^2 - \frac{i}{2\cosh \pi\mu}, \quad z \in \mathbb{C}\,,\ 0 < \mu < \infty\,. \tag{6.113}$$

When μ runs over the interval $[0, \infty)$, the complex number $\dfrac{i}{2\cosh \pi\mu}$, which appears in the right hand side of the equality (6.113), fill the interval $(0, i/2]$. Therefore

$$\inf_{\mu \in (0,\infty)} |D(z, \mu)| = \operatorname{dist}(z^2\,,\ [0, i/2]\,) \tag{6.114}$$

In particular,

$$\Big(\inf_{\mu\in(0,\infty)}|D(z,\mu)|>0\Big)\Leftrightarrow\big(\,z^2\notin[0,i/2]\,\big)\,,$$

or, what is the same,

$$\Big(\inf_{\mu\in(0,\infty)}|D(z,\mu)|>0\Big)\Leftrightarrow\Big(\,z\notin\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]\,\Big)\,, \qquad (6.115)$$

Thus the operator $M(\mathcal{L})=z\mathcal{I}-\mathfrak{F}_E$ is invertible if and only if the condition $z\notin\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]$ holds. The parts of statements 1 and 5 related to the operator $\mathfrak{F}_E$ are proved. The parts related to the adjoint operator $\mathfrak{F}_E^*$ can be proved analogously.

Moreover, if $z\notin\sigma_{\mathfrak{F}_E}$, then

$$(z\mathfrak{I}-\mathfrak{F}_E)^{-1}=(zI-F(\,.\,))^{-1}(\mathcal{L})\,, \qquad (6.116)$$

where the explicit expression the matrix $(zI-F(\mu))^{-1}$ is:

$$(zI-F(\mu))^{-1}=D(z,\mu)^{-1}\begin{bmatrix} z & f_{-+}(\mu)\\ f_{+-}(\mu) & z\end{bmatrix}\,. \qquad (6.117)$$

If $z\neq0$ belongs to the interval $\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]$, then the determinant $D(z,\mu)$ vanishes precisely at one point $\mu=\mu(z)\in[0,\infty)$. Therefore the matrix $zI-F(\mu)$ is invertible for every $\mu\in[0,\infty)$, $\mu\neq\mu(z)$, in particular it is invertible almost everywhere on $(0,\infty)$. Moreover, $\|(zI-F(\mu))^{-1}\|\to\infty$ as $\mu\to\mu(z)$, $\mu\neq\mu(z)$. Thus, the assertions 2 and 3 of Theorem 6.7 are consequences of Theorem 6.6.

If the image of the operator $z\mathfrak{I}-\mathfrak{F}_E$ is not dense in $L^2((0,\infty))$, then the number $\overline{z}$ is an eigenvalue of the adjoint operator $\mathfrak{F}_E^*$. However the operator $\mathfrak{F}_E^*$ has no eigenvalues. $\square$

Let us estimate the growth of the resolvent $(z\mathfrak{I}-\mathfrak{F}_E)^{-1}$ of the operator $\mathfrak{F}_E$ when z approaches the spectrum $\sigma_{\mathfrak{F}_E}$ of this operator. The inequality (6.111), together with the equality (6.114), give a two-sided estimate for the norm of the resolvent in terms of the value $\operatorname{dist}(z^2\,,\,[0,i/2])$:

$$\big\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\big\|\leq\frac{\big(2|z|^2+1\big)^{1/2}}{\operatorname{dist}(z^2\,,\,[0,i/2])}\,, \qquad (6.118a)$$

$$\frac{\big(2|z|^2+1\big)^{1/2}}{\operatorname{dist}(z^2\,,\,[0,i/2])}\sqrt{1-\frac{2\operatorname{dist}^2(z^2\,,[0,i/2])}{\big(2|z|^2+1\big)^2}}\leq\big\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\big\|\,.$$

The value under the square root in (6.118b) is positive since

$$\frac{2\mathrm{dist}^2(z^2\,,\,[0,i/2])}{\big(2|z|^2+1\big)^2} \leq \frac{2|z|^2}{\big(2|z|^2+1\big)^2} \leq \frac{1}{2}\,.$$

Since $(1-\alpha) \leq \sqrt{1-\alpha}$ for $0 \leq \alpha \leq 1$, then

$$1 - \frac{2\mathrm{dist}^2(z^2\,,\,[0,i/2])}{\big(2|z|^2+1\big)^2} \leq \sqrt{1 - \frac{2\mathrm{dist}^2(z^2\,,\,[0,i/2])}{\big(2|z|^2+1\big)^2}}\,.$$

Thus, the lower estimate for the norm of resolvent is

$$\frac{\big(2|z|^2+1\big)^{1/2}}{\mathrm{dist}(z^2\,,\,[0,i/2])} - \frac{2\mathrm{dist}(z^2\,,\,[0,i/2])}{\big(2|z|^2+1\big)^{3/2}} \leq \big\|(z\mathfrak{I}-\mathcal{F}_E)^{-1}\big\|\,. \quad (6.118b)$$

If the value $\mathrm{dist}(z^2\,,\,[0,i/2])$ is very small, then the lower estimate (6.118b) and the upper estimates (6.118a) are very close.

However we would like to estimate the norm of the resolvent $(z\mathfrak{I}-\mathcal{F}_E)^{-1}$ in terms of $|z-\zeta|$, where ζ is a point of the spectrum $\sigma_{\mathcal{F}_E}$ and z approaches ζ along the normal to the interval $\sigma_{\mathcal{F}_E}$. So we have to relate the values $\mathrm{dist}(z^2\,,\,[0,i/2])$ and $|z-\zeta|$, where $\zeta \in \sigma_{\mathcal{F}_E}$, and the interval $[z\,,\,\zeta]$ is orthogonal to the interval $\sigma_{\mathcal{F}_E}$.

Lemma 6.4. *Let ζ be a point of the spectrum $\sigma_{\mathcal{F}_E}$ of the operator $\mathcal{F}_E$:*

$$\zeta \in \Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]\,, \quad (6.119)$$

and the point z lies on the normal to the interval $\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]$ at the point ζ:

$$z = \zeta \pm |z-\zeta|e^{i3\pi/4}\,. \quad (6.120)$$

Then

$$\mathrm{dist}\big(z^2\,,\,[\,0\,,i/2]\,\big) \;=\; \begin{cases} 2|\zeta|\,|z-\zeta|\,, & \text{if } \;|z-\zeta| \leq |\zeta|\,,\\ |\zeta|^2+|z-\zeta|^2| \;=\; |z|^2, & \text{if } \;|z-\zeta| \geq |\zeta|\,. \end{cases} \quad (6.121)$$

Proof. The condition (6.119) means that $\zeta = \pm|\zeta|e^{i\pi/4}$. Substituting this expression for ζ into (6.120), we obtain

$$z^2 = \pm 2|\zeta|\,|z-\zeta| + i(|\zeta|^2 - |z-\zeta|^2)\,.$$

If $|z-\zeta|\leq|\zeta|$, then the point $|\zeta|^2-|z-\zeta|^2$ lies on the interval $[0,i/2]$. In this case, $\operatorname{dist}[0,i/2]=2|\zeta|\,|z-\zeta|$. If $|z-\zeta|\geq|\zeta|$, then the point $|\zeta|^2-|z-\zeta|^2$ lies on the half-axis $[0,-i\infty)$. In this case,

$$\operatorname{dist}(z^2\,,\,[0,\,,\,i/2]))=\sqrt{\left(|\zeta|^2-|z-\zeta|^2\right)^2+4|\zeta|^2|z-\zeta|^2}= \\ =|\zeta|^2+|z-\zeta|^2\,.$$

Since $|\zeta|^2+|z-\zeta|^2\geq 2|\zeta||z-\zeta|$, in any case the inequality

$$\operatorname{dist}(z^2\,,\,[0,,\,i/2]))\geq 2|\zeta|\,|z-\zeta|\,. \tag{6.122}$$

holds. □

Theorem 6.8. *Let ζ be a point of the spectrum $\sigma_{\mathfrak{F}_E}$ of the operator $\mathfrak{F}_E$, and let the point z lie on the normal to the interval $\sigma_{\mathfrak{F}_E}$ at the point ζ.*
Then

1. *The resolvent $(z\mathfrak{I}-\mathfrak{F}_E)^{-1}$ admits the estimate from above:*

$$\left\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\right\|\leq A(z)\frac{1}{|\zeta|}\cdot\frac{1}{|z-\zeta|}\,, \tag{6.123}$$

where $A(z)=\frac{(2|z|^2+1)^{1/2}}{2}$.

2. *If moreover the condition $|z-\zeta|\leq|\zeta|$ is satisfied, then the resolvent $(z\mathfrak{I}-\mathfrak{F}_E)^{-1}$ also admits the estimate from below:*

$$A(z)\frac{1}{|\zeta|}\cdot\frac{1}{|z-\zeta|}-B(z)|\zeta||z-\zeta|\leq\left\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\right\|, \tag{6.124}$$

where $A(z)$ is the same that in (6.123) and $B(z)=\frac{4}{(2|z|^2+1)^{3/2}}$.

3. *For $\zeta=0$, then the resolvent $(z\mathfrak{I}-\mathfrak{F}_E)^{-1}$ admits the estimates*

$$2A(z)\frac{1}{|z|^2}-B(z)\leq\left\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\right\|\leq 2A(z)\frac{1}{|z|^2}, \tag{6.125}$$

where $A(z)$ and $B(z)$ are the same that in (6.123), (6.124), and z is an arbitrary point of the normal.

In particular, if $\zeta\neq 0$, and z tends to ζ along the normal to the interval $\sigma_{\mathfrak{F}_E}$, then

$$\left\|(z\mathfrak{I}-\mathfrak{F}_E)^{-1}\right\|=\frac{A(\zeta)}{|\zeta|}\,\frac{1}{|z-\zeta|}+O(1)\,. \tag{6.126}$$

If $\zeta = 0$ and z tends to ζ along the normal to the interval $\sigma_{\mathcal{F}_E}$, then

$$\left\|(z\mathcal{I} - \mathcal{F}_E)^{-1}\right\| = |z|^{-2} + O(1)\,, \tag{6.127}$$

where $O(1)$ is a value which remains bounded as z tends to ζ.

Proof. The proof is based on the estimates (6.118) for the resolvent and on Lemma 6.4. Combining the inequality (6.122) with the estimate (6.118a), we obtain the estimate (6.123), which holds for *all* z lying on the normal to the interval $\sigma_{\mathcal{F}_E}$ at the point ζ. *If moreover z is close enough to z*, namely the condition $|z - \zeta| \leq |\zeta|$ is satisfied, then the equality holds in (6.122). Combining *the equality* (6.122) with the estimate (6.118b), we obtain the estimate (6.124).

The asymptotic relation (6.126) is a consequence of the inequalities (6.123) and (6.124) since $\frac{|A(z)-A(\zeta)|}{|z-\zeta|} = O(1)$ as z tends to ζ.

The asymptotic relation (6.127) is a consequence of the inequalities (6.118) and the equality $\mathrm{dist}\,(z^2\,,\,[0\,,\,i/2]) = |z|^2$ which holds for all z lying on the normal to the interval $\sigma_{\mathcal{F}_E}$ at the point $\zeta = 0$. (See (6.121) for $\zeta = 0$.) □

Remark 6.8. *The estimates* (6.123) *and* (6.124) *are formally true also for $\zeta = 0$, but in this case they are not rich in content.*

Corollary 6.2. *From the asymptotic relations* (6.126) *and* (6.127) *it follows that the operator $\mathcal{F}_E$ is not similar to a normal operator. Were the operator $\mathcal{F}_E$ similar to a normal operator $\mathcal{N}$, the resolvent $(z\mathcal{I} - \mathcal{F}_E)^{-1}$ would admit the estimate $\left\|(z\mathcal{I} - \mathcal{F}_E)^{-1}\right\| \leq C(\mathcal{N})\,\mathrm{dist}\,(z, \sigma_{\mathcal{F}_E})$, where $C(\mathcal{N}) < \infty$ is a constant which does not depend on z. However, this estimate is not compatible with the asymptotic relations* (6.126)*,* (6.127)*.*

Remark 6.9. The reasoning which was used in the proof of Theorem 6.7 says that the spectrum $\sigma_{\mathcal{F}_E}$ of the truncated Fourier operator $\mathcal{F}_E$ consists of the union of eigenvalues of the matrix $F(\mu)$, where the union is taken over all $\mu \in [0, \infty)$, and of the point $z = 0$. For each μ, the matrix $F(\mu)$ has two eigenvalues $\zeta_+(\mu)$ and $\zeta_-(\mu)$:

$$\zeta_+(\mu) = \zeta(\mu), \quad \zeta_-(\mu) = -\zeta(\mu)\,, \tag{6.128a}$$

where

$$\zeta(\mu) = e^{i\pi/4}\,\frac{1}{\sqrt{2\cosh \pi\mu}}\,. \tag{6.128b}$$

These eigenvalues are different. When μ runs over the interval $[0,\infty)$, the points $\zeta_+(\mu)$ fill the interval $(0\,,\,e^{i\pi/4}]$ and the points $\zeta_-(\mu)$ fill the interval $[-e^{i\pi/4}\,,\,0)$. Thus, the spectrum $\sigma_{\mathcal{F}_E}$ of the operator $\mathcal{F}_E$ splits naturally into the union

$$\sigma_{\mathcal{F}_E} = \sigma^+_{\mathcal{F}_E} \cup \sigma^-_{\mathcal{F}_E} \cup \{0\}, \tag{6.129}$$

where

$$\sigma^+_{\mathcal{F}_E} = \Big(0,\, \frac{1}{\sqrt{2}}\, e^{i\pi/4}\Big], \quad \sigma^-_{\mathcal{F}_E} = \Big[-\frac{1}{\sqrt{2}}\, e^{i\pi/4},\, 0\Big)\,. \tag{6.130}$$

So, *every point $\zeta \in \sigma_{\mathcal{F}_E} \setminus 0$ can be enumerated by a double index $(\mu,\, s)$, where $\mu \in [0,\infty)$, and s equals either $+$ or $-$:*

$$\zeta = \zeta_s(\mu), \quad \mu \in [0,\infty), \quad s \in \{+,\, -\}. \tag{6.131}$$

The correspondence between the set $\sigma_{\mathcal{F}_E} \setminus 0$ and the above set of double indices is one-to-one.

5. Let us develop an operator calculus for the truncated Fourier operator $\mathcal{F}_E$, $E = (0,\infty)$. The starting point is the representation of the resolvent of the operator $\mathcal{F}_E$. The resolvent $(z\mathfrak{I} - \mathcal{F}_E)^{-1}$ of the operator $\mathcal{F}_E$ is the integral operator with the kernel $K_{(z\mathfrak{I}-\mathcal{F}_E)^{-1}}(\xi,t)$:

$$K_{(z\mathfrak{I}-\mathcal{F}_E)^{-1}}(\xi,t) = \int\limits_{t\in(0,\infty)} \psi(\xi,\,\mu)^*(zI - F(\mu))^{-1}\psi(t,\,\mu)\,\frac{d\mu}{2\pi}\,, \tag{6.132}$$

where the expression for the matrix $(zI - F(\mu))^{-1}$ is presented in (6.117), (6.113). (The expression (6.132) is the content of the Statement 5 of Theorem 6.7.) Let $h(z)$ be a function which is holomorphic in a neighborhood of the interval $\Big[-\frac{1}{\sqrt{2}}\, e^{i\pi/4},\, \frac{1}{\sqrt{2}}\, e^{i\pi/4}\Big]$. (This interval is the spectrum of the operator $\mathcal{F}_E$.) According to the traditional functional calculus, the operator $h(\mathcal{F}_E)$ is defined as

$$h(\mathcal{F}_E) = \frac{1}{2\pi i}\int\limits_{\Gamma} h(z)(z\mathfrak{I} - \mathcal{F}_E)^{-1}dz\,, \tag{6.133}$$

where Γ is a closed simple contour which is contained in the holomorphy domain of the function h, includes the spectrum of $\mathcal{F}_E$ and is turned counterclockwise. Substituting the expression (6.132) for the

resolvent $(z\mathfrak{I}-\mathcal{F}_E)^{-1}$ and changing order of integration, we obtain the following expression for the kernel $K_{h(\mathcal{F}_E))}$ of the operator $h(\mathcal{F}_E))$:

$$K_{h(\mathcal{F}_E))} = \int\limits_{\mu\in(0,\infty)} \psi(\xi,\,\mu)^* h(F(\mu))\psi(t,\,\mu)\,\frac{d\mu}{2\pi}\,, \tag{6.134}$$

$$h(\mathcal{F}_E)\,x(t) = \int\limits_{\xi\in(0,\infty)} x(\xi) K_{h(\mathcal{F}_E))}(\xi,\,t)\,d\xi\,, \tag{6.135}$$

where for every $\mu\in(0,\infty)$,

$$h(F(\mu)) = \frac{1}{2\pi i}\int\limits_{\Gamma} h(z)(zI-F(\mu))^{-1}dz\,, \tag{6.136}$$

and $F(\mu)$ is the matrix which appears in (6.59). The integral in (6.136) can be calculated using the expression (6.117), (6.113) for the matrix function $(zI-F(\mu))^{-1}$.

According to (6.113),

$$D(z,\,\mu) = (z-\zeta(\mu))(z+\zeta(\mu))\,,$$

where $\zeta(\mu)$ was defined in (6.128b). The matrix function $(zI-F(\mu))^{-1}$ is a rational function of z. The only singularities of this function inside Γ are the simple poles at the points $z=\zeta(\mu)$ and $z=-\zeta(\mu)$. Calculating the integral in (6.136), we obtain

$$h(F(\mu)) = \begin{bmatrix} \dfrac{h(\zeta(\mu))+h(-\zeta(\mu))}{2} & \dfrac{h(\zeta(\mu))-h(-\zeta(\mu))}{2\zeta(\mu)}f_{-+}(\mu) \\ \dfrac{h(\zeta(\mu))-h(-\zeta(\mu))}{2\zeta(\mu)}f_{+-}(\mu) & \dfrac{h(\zeta(\mu))+h(-\zeta(\mu))}{2} \end{bmatrix} \tag{6.137}$$

Thus, if $h(z)$ is a function holomorphic on the spectrum of the operator $\mathcal{F}_E$, then the operator $h(\mathcal{F}_E)$ can be expressed in the form (6.134), where the matrix function $h(F(\mu))$ is expressed by (6.137).

However, the expression (6.137) has a meaning not only for functions h which are holomorphic on the spectrum of $\mathcal{F}_E$, i.e. on the interval $\Big[-\frac{1}{\sqrt{2}}\,e^{i\pi/4},\,\frac{1}{\sqrt{2}}\,e^{i\pi/4}\Big]$. *Given μ, the expression* (6.137) *is meaningful for any function h which is defined on the two point set $\zeta(\mu)$ and $-\zeta(\mu)$*, which constitute the spectrum of 2×2 matrix $F(\mu)$.

To define a function of a 2×2 matrix with different eigenvalues, it is enough to know the values of this function at the eigenvalues of this matrix only.

The formula (6.137) gives the expression for $h(F(\mu))$ in the sense of the elementary functional calculus of matrices. Indeed, let

$$E_+(\mu) = \begin{bmatrix} \frac{1}{2} & \frac{f_{-+}(\mu)}{2\zeta(\mu)} \\ \frac{f_{+-}(\mu)}{2\zeta(\mu)} & \frac{1}{2} \end{bmatrix}, \quad E_-(\mu) = \begin{bmatrix} \frac{1}{2} & -\frac{f_{-+}(\mu)}{2\zeta(\mu)} \\ -\frac{f_{+-}(\mu)}{2\zeta(\mu)} & \frac{1}{2} \end{bmatrix}. \tag{6.138}$$

(The expressions for E_+ and E_- can be obtained from (6.138) if we set $h(\zeta(\mu)) = 1$, $h(-\zeta(\mu)) = 0$ and $h(\zeta(\mu)) = 0$, $h(-\zeta(\mu)) = 1$ respectively). We note that according to (6.58) and (6.63),

$$\zeta^2(\mu) = f_{+-}(\mu) f_{-+}(\mu)\,. \tag{6.139}$$

Using (6.139) we verify that

$$E_+(\mu)^2 = E_+(\mu),\;\; E_-(\mu)^2 = E_-(\mu),\;\; E_+(\mu)E_-(\mu) = E_-(\mu)E_+(\mu) = 0\,. \tag{6.140}$$

The equality

$$E_+(\mu) + E_-(\mu) = I \tag{6.141}$$

is evident. The matrix $F(\mu)$ is representable as

$$F(\mu) = \zeta_+(\mu)E_1(\mu) + \zeta_-(\mu)E_-(\mu)\,. \tag{6.142}$$

In other words, the matrices $E_+(\mu)\,, E_-(\mu)$ are the spectral projectors of the matrix $F(\mu)$ onto its eigenspaces corresponding to the eigenvalues *the numbers $\zeta_+(\mu)$ and $\zeta_-(\mu)$ respectively.*

The complex conjugated numbers $\overline{\zeta_+(\mu)}$, $\overline{\zeta_-(\mu)}$ are the eigenvalues of the hermitian conjugated matrix $F^*(\mu)$ and the hermitian conjugated matrices $E_+^*(\mu)$, $E_-^*(\mu)$,

$$E_+^*(\mu) = \begin{bmatrix} \frac{1}{2} & \frac{\overline{f_{+-}(\mu)}}{2\overline{\zeta(\mu)}} \\ \frac{\overline{f_{-+}(\mu)}}{2\overline{\zeta(\mu)}} & \frac{1}{2} \end{bmatrix}, \quad E_-^*(\mu) = \begin{bmatrix} \frac{1}{2} & -\frac{\overline{f_{+-}(\mu)}}{2\overline{\zeta(\mu)}} \\ -\frac{\overline{f_{-+}(\mu)}}{2\overline{\zeta(\mu)}} & \frac{1}{2} \end{bmatrix}, \tag{6.143}$$

are the spectral projectors onto the eigenspaces of the matrix $F^*(\mu)$ corresponding to these eigenvalues:

$$E_+^*(\mu)^2 = E_+^*(\mu),\;\; E_-^*(\mu)^2 = E_-^*(\mu),\; E_+^*(\mu)E_-^*(\mu) = E_-^*(\mu)E_+^*(\mu) = 0\,, \tag{6.144}$$

$$E_+^*(\mu) + E_-^*(\mu) = I\,, \tag{6.145}$$

$$F^*(\mu) = \overline{\zeta_+(\mu)}\, E_+^*(\mu) + \overline{\zeta_-(\mu)}\, E_-^*(\mu)\,. \tag{6.146}$$

Definition 6.7. *If h is the function taking the values $h(\zeta_+(\mu))$ and $h(\zeta_-(\mu))$ at the points $\zeta_+(\mu)$, $\zeta_-(\mu)$, (6.128a), then the matrix $h(F(\mu))$ is defined as*

$$h(F(\mu)) \stackrel{\text{def}}{=} h(\zeta_+(\mu))E_+(\mu) + h(\zeta_-(\mu))E_-(\mu)\,. \tag{6.147a}$$

If h is the function taking the values $h(\overline{\zeta_+(\mu)})$ and $h(\overline{\zeta_-(\mu)})$ at the points $\overline{\zeta_+(\mu)}$, $\overline{\zeta_-(\mu)}$, then the matrix $h(F^*(\mu))$ is defined as

$$h(F^*(\mu)) \stackrel{\text{def}}{=} h(\overline{\zeta_+(\mu)})E_+^*(\mu) + h(\overline{\zeta_-(\mu)})E_-^*(\mu)\,. \tag{6.147b}$$

The definition (6.147a) agrees with (6.137).

Definition 6.8. *If $h(\zeta)$ is a complex-valued function defined on a subset S of the complex plane, than the* conjugated *function $\overline{h}(\zeta)$ is a function defined on the conjugated set $\overline{S}$ by the equality*

$$\overline{h}(\zeta) \stackrel{\text{def}}{=} \overline{h(\overline{\zeta})}\,. \tag{6.148}$$

Let us estimate the norm of the matrix $h(F(\mu))$ in terms of the values $h(\pm\zeta(\mu))$. Applying this estimate of Lemma 6.3 to the matrix $h(F(\mu))$, (6.137), and taking into account the estimates (6.65), we obtain

$$\begin{aligned} \tfrac{1}{8}\bigg(|h(\zeta(\mu)) + h(-\zeta(\mu))| + \frac{|h(\zeta(\mu)) - h(-\zeta(\mu))}{|\zeta(\mu)|}\bigg) &\leq \\ \leq \|h(F(\mu))\| &\leq \\ \leq |h(\zeta(\mu)) + h(-\zeta(\mu))| &+ \frac{|h(\zeta(\mu)) - h(-\zeta(\mu))}{|\zeta(\mu)|} \end{aligned}$$

Since $|\zeta(\mu)| < 1$,

$$|h(\zeta(\mu))| + |h(-\zeta(\mu))| \leq |h(\zeta(\mu)) + h(-\zeta(\mu))| + \frac{|h(\zeta(\mu)) - h(-\zeta(\mu))}{|\zeta(\mu)|}.$$

Thus, the following two-sided estimate for the horm of the matrix $h(F(\mu))$ is obtained:

Lemma 6.5. *For $\mu \in (0, \infty)$, let the matrix $h(F(\mu))$ be defined by* (6.147) *with $\zeta_{1,2}(\mu)$ from* (6.128a)*. Then the following estimate holds:*

$$1/16 \left(\max\big(|h(\zeta(\mu))|, |h(-\zeta(\mu))|\big) + \frac{|h(\zeta(\mu)) - h(-\zeta(\mu))}{|\zeta(\mu)|} \right) \leq$$
$$\leq \|h(F(\mu))\| \leq$$
$$\leq 2 \left(\max\big(|h(\zeta(\mu))|, |h(-\zeta(\mu))|\big) + \frac{|h(\zeta(\mu)) - h(-\zeta(\mu))}{|\zeta(\mu)|} \right). \quad (6.149)$$

Definition 6.9. *A function $h(\zeta)$ is said to be $\mathcal{F}_E$-*admissible *if*

1. *$h(\zeta)$ is measurable and defined almost everywhere with respect to the Lebesgue measure on the interval $\left[-\frac{1}{\sqrt{2}}\, e^{i\pi/4},\, \frac{1}{\sqrt{2}}\, e^{i\pi/4}\right]$.*
2. *The norm $\|h\|_{\mathcal{F}_E}$ is finite, where*

$$\|h\|_{\mathcal{F}_E} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta\in\sigma_{\mathcal{F}_E}} |h(\zeta)| + \operatorname*{ess\,sup}_{\zeta\in\sigma_{\mathcal{F}_E}} \frac{|h(\zeta) - h(-\zeta)|}{|\zeta|}\,, \qquad (6.150)$$

and $\sigma_{\mathcal{F}_E} = \left[-\frac{1}{\sqrt{2}}\, e^{i\pi/4},\, \frac{1}{\sqrt{2}}\, e^{i\pi/4}\right]$ is the spectrum of the operator $\mathcal{F}_E$.

An analogous definition related to the adjoint operator $\mathcal{F}_E^*$ is:

Definition 6.10. *A function $h(\zeta)$ is said to be $\mathcal{F}_E^*$-*admissible *if*

1. *$h(\zeta)$ is measurable and defined almost everywhere with respect to the Lebesgue measure on the interval $\left[-\frac{1}{\sqrt{2}}\, e^{-i\pi/4},\, \frac{1}{\sqrt{2}}\, e^{-i\pi/4}\right]$.*
2. *The norm $\|h\|_{\mathcal{F}_E^*}$ is finite, where*

$$\|h\|_{\mathcal{F}_E^*} \overset{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta\in\sigma_{\mathcal{F}_E^*}} |h(\zeta)| + \operatorname*{ess\,sup}_{\zeta\in\sigma_{\mathcal{F}_E^*}} \frac{|h(\zeta) - h(-\zeta)|}{|\zeta|}\,, \qquad (6.151)$$

and $\sigma_{\mathcal{F}_E^*} = \left[-\frac{1}{\sqrt{2}}\, e^{-i\pi/4},\, \frac{1}{\sqrt{2}}\, e^{-i\pi/4}\right]$ is the spectrum of the operator $\mathcal{F}_E^*$.

Definition 6.11. *The set of all $\mathcal{F}_E$-*admissible *functions provided by natural "pointwise" algebraic operation and the norm* (6.150) *is denoted by $\mathfrak{B}_{\mathcal{F}_E}$.*

*The set of all $\mathcal{F}_E^*$-*admissible *functions provided by natural "pointwise" algebraic operation and the norm* (6.151) *is denoted by $\mathfrak{B}_{\mathcal{F}_E^*}$.*

It is clear that each of the spaces $\mathfrak{B}_{\mathcal{F}_E}$, $\mathfrak{B}_{\mathcal{F}_E^*}$ is a Banach algebra, and

$$\|h_1 h_2\|_{\mathcal{F}_E} \leq \|h_1\|_{\mathcal{F}_E} \|h_2\|_{\mathcal{F}_E} \quad \text{or} \quad \|h_1 h_2\|_{\mathcal{F}_E^*} \leq \|h_1\|_{\mathcal{F}_E^*} \|h_2\|_{\mathcal{F}_E^*} \tag{6.152}$$

for every $h_1,\, h_2 \in \mathfrak{B}_{\mathcal{F}_E}$ or $h_1,\, h_2 \in \mathfrak{B}_{\mathcal{F}*_E}$ respectively.

Lemma 6.6. *Let $x(\zeta)$ be a function belonging to the algebra $\mathfrak{B}_{\mathcal{F}_E}$.*

The function $x(\zeta)$ is invertible (in $\mathfrak{B}_{\mathcal{F}_E}$) if and only if the set of its values is separated from zero, that is the following condition

$$\operatorname*{ess\,inf}_{\zeta \in \sigma_{\mathcal{F}_E}} |x(\zeta)| > 0\,. \tag{6.153}$$

holds.

If the condition (6.153) *holds, then the inverse function $x^{-1}(\zeta)$ is:*

$$x^{-1}(\zeta) = (x(\zeta))^{-1}, \quad \zeta \in \sigma_{\mathcal{F}_E}\,, \tag{6.154}$$

and

$$\|x^{-1}(\zeta)\|_{\mathcal{F}_E} \leq \|x(\zeta)\|_{\mathcal{F}_E} \cdot \Big(\operatorname*{ess\,inf}_{\zeta \in \sigma_{\mathcal{F}_E}} |x(\zeta)|\Big)^{-2}. \tag{6.155}$$

The inequality (6.149) implies the inequality

$$\tfrac{1}{16}\|h\|_{\mathcal{F}_E} \leq \operatorname*{ess\,sup}_{\mu \in (0,\infty)} \|h(\mathcal{F}_E)(\mu)\| \leq 2\|h\|_{\mathcal{F}_E}\,. \tag{6.156}$$

Lemma 6.7. *A function h belongs to the algebra $\mathfrak{B}_{\mathcal{F}_E}$ if and only if the conjugate function $\overline{h}$ belongs to the algebra $\mathfrak{B}_{\mathcal{F}_E^*}$. Moreover,*

$$\|h\|_{\mathfrak{B}_{\mathcal{F}_E}} = \|\overline{h}\|_{\mathfrak{B}_{\mathcal{F}_E^*}}$$

Now we are in position to define the operators $h(\mathcal{F}_E)$ and $h(\mathcal{F}_E^*)$, where h is an arbitrary function belonging to the algebra $\mathfrak{B}_{\mathcal{F}_E}$ or $\mathfrak{B}_{\mathcal{F}_E^*}$ respectively.

Definition 6.12. *For a function h belonging to the algebra $\mathfrak{B}_{\mathcal{F}_E}$, the operator $h(\mathcal{F}_E)$ is defined as*

$$(h(\mathcal{F}_E)x)(t) = \int\limits_{\mu \in (0,\infty)} \hat{x}(\mu) h(F(\mu)) \psi(t,\mu) \frac{d\mu}{2\pi}, \quad \forall\, x \in L^2(0,\infty)\,, \tag{6.157a}$$

where $h(F(\mu))$ is defined by (6.137), *and $\hat{x}$ is defined by* (6.37).

$$(h(\mathfrak{F}_E^*)x)(t) = \int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)h(F^*(\mu))\psi(t,\mu)\frac{d\mu}{2\pi}, \quad \forall\, x \in L^2(0,\infty)\,, \tag{6.157b}$$

where $h(F^*(\mu))$ is defined by (6.147b), *and $\hat{x}$ is defined by* (6.37).

In other words, the operator $h(\mathfrak{F}_E)$ is defined as the matrix function $M_h(\mathcal{L})$ of the operator $\mathcal{L}$ in the sense of Definition 6.5, *where the matrix function $M_h(\mu) = h(F(\mu))$ is defined in* (6.147a)-(6.138)-(6.128a)-(6.128b).

From the inequalities (6.156) and Theorem 6.4 it follows the two-sided estimate for the norm of the operator $h(\mathfrak{F}_E)$:

$$\tfrac{1}{16}\|h\|_{\mathfrak{F}_E} \leq \|(h(\mathfrak{F}_E)\| \leq 2\|h\|_{\mathfrak{F}_E}\,. \tag{6.158}$$

Theorem 6.9. *The mapping*

$$h(\zeta) \to h(\mathfrak{F}_E) \tag{6.159}$$

defined by (6.158) (*see* Definition 6.12) *is a homomorphism of the algebra $\mathfrak{B}_{\mathfrak{F}_E}$ of $\mathfrak{F}_E$-admissible functions into the algebra of bounded operators in $L^2((0,\infty))$:*

1. *If $h(\zeta) = 1$, then $h(\mathfrak{F}_E) = \mathfrak{I}$.*
2. *If $h(\zeta) = \zeta$, then $h(\mathfrak{F}_E) = \mathfrak{F}_E$.*
3. *If $h(\zeta) = \alpha_1 h_1(\zeta) + \alpha_2 h_2(\zeta)$, where α_1, α_2 are complex numbers and h_1, $h_2 \in \mathfrak{B}_{\mathfrak{F}_E}$, then*
$$h(\mathfrak{F}_E) = \alpha_1 h_1(\mathfrak{F}_E) + \alpha_2 h_2(\mathfrak{F}_E)\,.$$
4. *If $h(\zeta) = h_1(\zeta)\cdot h_2(\zeta)$, where h_1, $h_2 \in \mathfrak{B}_{\mathfrak{F}_E}$, then*
$$h(\mathfrak{F}_E) = h_1(\mathfrak{F}_E)\, h_2(\mathfrak{F}_E)\,.$$
5. *If $h \in \mathfrak{B}_{\mathfrak{F}_E}$, then the operator $h(\mathfrak{F}_E)$ is an invertible operator in $L^2((0,\infty))$ if and only if the function h is invertible in the algebra $\mathfrak{B}_{\mathfrak{F}_E}$. If the function h is invertible, and $h^{-1}(\zeta)$ is the function inverse to h, then*
$$(h(\mathfrak{F}_E))^{-1} = h^{-1}(\mathfrak{F}_E)\,.$$

6. *If $h \in \mathfrak{B}_{\mathcal{F}_E}$, then the operator $(h(\mathcal{F}_E))^*$, conjugated to the operator $\mathcal{F}_E$, cat be expressed as the conjugated function $\overline{h}$ of the operator $\mathcal{F}_E^*$:*

$$(h(\mathcal{F}_E))^* = \overline{h}(\mathcal{F}_E^*)\,. \tag{6.160}$$

Proof. If $h(\zeta) = 1$, then $M_h(\mu) = E_1(\mu) + E_2(\mu) = I$. According to Statement 1 of Theorem 6.5, $M_h((L)) = I$.

If $h(\zeta) = \zeta$, then $M_h(\mu) = F(\mu)$, (6.142), and according to Remark 6.4, $M_h(\mathcal{L}) = F(\mathcal{L}) = \mathcal{F}_E$.

The statements 3 and 4 of the Theorem are direct consequences of properties of the mapping $M(\mu) \to M(\mathcal{L})$, which are stated in Theorem 6.5, and of properties of the mapping $h(\zeta) \to M_h(\mu) = h(F(\mu))$ defined in (6.147). In particular, $M_{h_1h_2}(\mu) = M_{h_2}(\mu)M_{h_1}(\mu)$. Thus $M_{h_1h_2}(\mathcal{L}) = M_{h_2}(\mathcal{L})M_{h_1}(\mathcal{L})$, i.e. $(h_1h_2)(\mathcal{F}_E) = h_1(\mathcal{F}_E)h_2(\mathcal{F}_E)$.

If the function h is invertible, then according to Statements 1 and 4 of Theorem, which already are proved,

$$h^{-1}(\mathcal{F}_E)h(\mathcal{F}_E) = h(\mathcal{F}_E)(h^{-1}(\mathcal{F}_E) = I.$$

If the function h is not invertible, then, according to Lemma 6.6,

$$\operatorname*{ess\,inf}_{\zeta\in\sigma_{\mathcal{F}_E}} |x(\zeta)| = 0\,.$$

This means that for every preassigned $\varepsilon > 0$, there exists the subset S of the spectrum $\sigma_{\mathcal{F}_E}$ with the properties

$$\operatorname{mes} S > 0, \quad |h(\zeta)| < \varepsilon \quad \forall\, \zeta \in S\,.$$

We construct the function $x \in L^2((0,\infty))$ such that

$$\|x\|_{L^2} = 1, \quad \|h(\mathcal{F}_E)x\|_{L^2} < \varepsilon\,.$$

Since ε can be taken arbitrary small, this means that the operator $h(\mathcal{F}_E)$ is not invertible. We construct the 1×2 vector-function $\hat{x}(\mu)$ rather the functions $x(t)$. Without loss of generality we can assume that either $S \subseteq \sigma^+_{\mathcal{F}_E}$, or $S \subseteq \sigma^-_{\mathcal{F}_E}$, where $\sigma^+_{\mathcal{F}_E}$, $\sigma^-_{\mathcal{F}_E}$ are defined in (6.130). At least one of the sets $S \cap \sigma^+_{\mathcal{F}_E}$ and $S \cap \sigma^-_{\mathcal{F}_E}$ is of positive measure.) For definiteness, let $S \subseteq \sigma^+_{\mathcal{F}_E}$. The function $\zeta(\mu)$ introduced in (6.128b) maps one-to-one the interval $[0,\infty)$ onto the set $\sigma^+_{\mathcal{F}_E}$.

Let us construct the function $\hat{x}(\mu)$. If $\mu \in (0,\infty)$ is such that $\zeta(\mu) \notin S$, then we set $\hat{x}(\mu) = 0$. If $\zeta(\mu) \in S$, then we choose $\hat{x}(\mu)$ such that the conditions

$$\hat{x}(\mu) \neq 0, \quad \hat{x}(\mu) = \hat{x}(\mu)E_+(\mu) \tag{6.161}$$

are fulfilled, where $E_+(\mu)$ os the projector of rank one introduced in (6.138). For each μ, the conditions (6.161) determine $\hat{x}(\mu)$ up to a scalar non-zero factor. We choose these factors in such a way that the function $\hat{x}(\mu)$ to be measurable and satisfy the condition

$$\int\limits_{\mu\in(0,\infty)} \hat{x}(\mu)\hat{x}^*(\mu)\frac{d\mu}{2\pi} = 1\,.$$

In view of (6.147a) and (6.161), $\hat{x}(\mu)h(F(\mu) = h(\zeta(\mu))\hat{x}(\mu)$, thus

$$\hat{x}(\mu)h(F(\mu)h(F(\mu))^*\hat{x}(\mu)^* \leq \varepsilon^2\hat{x}(\mu)\hat{x}^*(\mu) \quad \forall\mu\in[0,\infty)\,.$$

The statement 5 is proved.

The statement 6 is a direct consequence of Definitions 6.12, 6.7, and 6.8.

□

6. After the functional calculus for the operator $\mathcal{F}_E$ was developed, we have tools to define and investigate spectral projectors related to the operator $\mathcal{F}_E$.

According (6.157), the functions $h(\mathcal{F}_E)$ and $h(\mathcal{F}_E^*)$ of the operators $\mathcal{F}_E$) and $\mathcal{F}_E^*$ are expressed in the form which can be considered as integral expansions in the eigenfunctions $\psi(t,\mu)$ of the operator $\mathcal{L}$. In particular, the expressions (6.83) for the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$ looks as integral expansions in the eigenfunctions of the operator $\mathcal{L}$. Now we are in position to present the expansions (6.157), and in particular the expansions (6.83), as *integral expansions in the eigenfunctions of the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$ themselves rather in the eigenfunctions of the operator $\mathcal{L}$.*

To do this, we have first of all to clarify what are eigenfunctions of the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$. These eigenfunctions are not L^2-functions but 'generalized' functions. According to general ideology, going back to A. Ja. Povzner [Pov1], [Pov2] and F. Mautner [Mau], generalized eigenfunctions of an operator can be obtained by differentiation of the resolution of identity related to this operator. Considerations of A. Ja. Povzner and F. Mautner and their followers are related to self-adjoint operators. Though the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$ are non self-adjoint, the functional calculus developed by us allows to some extent to work with these operators as if they are self-adjoint. In particular we construct objects which may be considered as resolutions of identity related to the operators $\mathcal{F}_E$ and $\mathcal{F}_E^*$. The resolution of identity

related to the operator $\mathcal{F}_E$ is a family of its spectral projectors. We construct this family of spectral projectors as the family of functions of the operator $\mathcal{F}_E$ which functions are the indicator functions of subsets of the spectrum $\sigma_{\mathcal{F}_E}$. Though this family of subsets is not so rich as in the case of self-adjoint operator and does not contain *all* Borelian subsets of $\sigma_{\mathcal{F}_E}$, it is rich enough for our goal.

Definition 6.13. *For a subset Δ of the complex plane, we define its* symmetric part Δ_s *and* asymmetric part Δ_a*:*

$$\Delta_s = \Delta \cap (-\Delta), \quad \Delta_a = \Delta \setminus (-\Delta)\,. \tag{6.162}$$

Here, as usual, $-\Delta = \{\, z \in \mathbb{C} : \, -z \in \Delta \,\}$. So,

$$\Delta = \Delta_s \cup \Delta_a, \quad \Delta_s \cap \Delta_a = \emptyset, \quad \Delta_s = -\Delta_s, \quad \Delta_a \cap (-\Delta_a) = \emptyset\,. \tag{6.163}$$

With every subset of $\Delta \in \mathbb{C}$, we associate its indicator function $\chi_{\Delta}(z)$:

$$\chi_{\Delta}(z) = 1 \ \text{ if } \ z \in \Delta, \ \chi_{\Delta}(z) = 0 \ \text{ if } \ z \notin \Delta\,.$$

Definition 6.14. *The set S, $S \subseteq \sigma_{\mathcal{F}_E}$ is* essentially separated from zero *if*

$$\operatorname{ess\,dist}(S, 0) > 0\,.$$

Lemma 6.8. *Let Δ be a subset of the spectrum $\sigma_{\mathcal{F}_E}$ of the operator $\mathcal{F}_E$. The indicator function χ_{Δ} belongs to the set $\mathfrak{B}_{\mathcal{F}_E}$ of $\mathcal{F}_E$-admissible functions if and only if the asymmetric part Δ_a of the set Δ is essentially separated from zero.*

Proof. Since the function $\chi_{\Delta}(\zeta)$ is bounded (either $|\chi_{\Delta}(\zeta)| = 1$ or $|\chi_{\Delta}(\zeta)| = 0$), the function $\chi_{\Delta}(\zeta)$ is $\mathcal{F}_E$-admissible if and only if the function $\dfrac{\chi_{\Delta}(\zeta) - \chi_{\Delta}(-\zeta)}{\zeta}$ is essentially bounded. In view of (6.163), $\chi_{\Delta}(\zeta) - \chi_{\Delta}(-\zeta) = \chi_{\Delta_a}(\zeta) - \chi_{\Delta_{-a}}(-\zeta)$. From the other hand,

$$|\chi_{\Delta_a}(\zeta) - \chi_{\Delta_{-a}}(-\zeta)| = \chi_{\Delta_a \cup (-\Delta_a)}(\zeta)\,,$$

so the function $\dfrac{\chi_{\Delta}(\zeta) - \chi_{\Delta}(-\zeta)}{\zeta}$ is essentially bounded if and only if the function $\dfrac{\chi_{\Delta_a \cup (-\Delta_a)}(\zeta)}{\zeta}$ is essentially bounded. The last function is essentially bounded if and only if the set $\Delta_a \cup (-\Delta_a)$ is essentially separated from zero. From the structure of the set $\Delta_a \cup (-\Delta_a)$ it is clear that the set $\Delta_a \cup (-\Delta_a)$ is essentially separated from zero if and only if the set Δ_a is essentially separated from zero. $\square$

Definition 6.15. *The Borelian subset* Δ *of the spectrum* $\sigma_{\mathcal{F}_E}$ *is said to be* $\mathcal{F}_E$*-admissible if the indicator function* $\chi_{\Delta}(\zeta)$ *is a* $\mathcal{F}_E$*-admissible function.*

Lemma 6.9. *Let* Δ_1 *and* Δ_2 *be* $\mathcal{F}_E$*-admissible subsets of the spectrum* $\mathcal{F}_E$*. Then the sets* $\Delta_1 \cup \Delta_2$*,* $\Delta_1 \cap \Delta_2$ *and* $\Delta_1 \setminus \Delta_2$ *are* $\mathcal{F}_E$*-admissible sets as well. In particular if the set* Δ *is admissible, its complement, the set* $\sigma_{\mathcal{F}_E} \setminus \Delta$*, is admissible as well.*

Definition 6.16. *Let* Δ *be a* $\mathcal{F}_E$*-admissible subset of the spectrum* $\mathcal{F}_E$*. The operator* $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ *is defined as*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) \stackrel{\text{def}}{=} \chi_{\Delta}(\mathcal{F}_E), \tag{6.164}$$

where $\chi_{\Delta}(\zeta)$ *is the indicator function of the set* Δ *and the function* $\chi_{\Delta}(\mathcal{F}_E)$ *of the operator* $\mathcal{F}_E$ *is understood in the sense of* Definition 6.12*.*

Remark 6.10. *According to* (6.147a)*,*

$$\chi_{\Delta}(F(\mu)) = \begin{cases} I, & if \;\; \zeta_+(\mu) \in \Delta,\, \zeta_-(\mu) \in \Delta; \\ E_+(\mu), & if \;\; \zeta_+(\mu) \in \Delta,\, \zeta_-(\mu) \notin \Delta; \\ E_-(\mu), & if \;\; \zeta_+(\mu) \notin \Delta,\, \zeta_-(\mu) \in \Delta; \\ 0, & if \;\; \zeta_+(\mu) \notin \Delta,\, \zeta_-(\mu) \notin \Delta\,. \end{cases} \tag{6.165}$$

Theorem 6.10. *The family of the operators* $\{\mathcal{P}_{\mathcal{F}_E}(\Delta)\}_{\Delta}$*, where* Δ *runs over the family of all* $\mathcal{F}_E$*-admissible sets, possesses the following properties:*

1. *If the sets* Δ_1 *and* Δ_2 *are* $\mathcal{F}_E$*-admissible, then*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta_1 \cap \Delta_2) = \mathcal{P}_{\mathcal{F}_E}(\Delta_1) \cdot \mathcal{P}_{\mathcal{F}_E}(\Delta_2); \tag{6.166}$$

In particular, for every $\mathcal{F}_E$*-admissible set* Δ*, the operator* $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ *is a projector:*

$$\mathcal{P}^2_{\mathcal{F}_E}(\Delta) = \mathcal{P}_{\mathcal{F}_E}(\Delta)\,; \tag{6.167}$$

2. *The projectors corresponding to the* $\mathcal{F}_E$*-admissible sets* $\emptyset$ *and* $\sigma_{\mathcal{F}_E}$ *are:*

$$\mathcal{P}_{\mathcal{F}_E}(\emptyset) = 0\,; \quad \mathcal{P}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}) = \mathcal{I}, \tag{6.168}$$

where $\mathcal{I}$ *is the identity operator in the space* $L^2(E)$*.*

3. *The correspondence* $\Delta \to \mathcal{P}_{\mathcal{F}_E}(\Delta)L^2(E)$ *between subsets of the spectrum* $\sigma_{\mathcal{F}_E}$ *and subspaces of the space* $L^2(E)$ *preserves the order:*

$$\text{If} \quad \Delta_1 \subset \Delta_2, \quad \text{then} \quad \mathcal{P}_{\mathcal{F}_E}(\Delta_1)L^2(E) \subseteq \mathcal{P}_{\mathcal{F}_E}(\Delta_2)L^2(E)\,. \tag{6.169}$$

4. *If the sets* Δ_1 *and* Δ_2 *are* $\mathcal{F}_E$*-admissible, and* $\Delta_1 \cap \Delta_2 = \emptyset$*, then*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta_1 \cup \Delta_2) = \mathcal{P}_{\mathcal{F}_E}(\Delta_1) + \mathcal{P}_{\mathcal{F}_E}(\Delta_2)\,; \tag{6.170}$$

In particular, for every $\mathcal{F}_E$*-admissible set* Δ*, the equality*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) + \mathcal{P}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus \Delta) = \mathcal{I}\,. \tag{6.171}$$

holds.

Proof. The mapping $\Delta \to \chi_\Delta(\zeta)$ possesses the properties:

$$\begin{gathered}
\chi_{\Delta_1 \cap \Delta_2}(\zeta) = \chi_{\Delta_1}(\zeta) \cdot \chi_{\Delta_2}(\zeta) \text{ for every } \Delta_1,\, \Delta_2,\\
\chi_{\Delta_1 \cup \Delta_2}(\zeta) = \chi_{\Delta_1}(\zeta) + \chi_{\Delta_2}(\zeta) \text{ if } \Delta_1 \cap \Delta_2 = \emptyset,\\
\chi_\emptyset(\zeta) \equiv 0, \text{ and } \chi_{\sigma_{\mathcal{F}_E}}(\zeta) = 1 \text{ for } \zeta \in \sigma_{\mathcal{F}_E}.
\end{gathered}$$

Statements 1 - 4 of the present Theorem are consequences of these properties of the mapping $\Delta \to \chi_\Delta(\zeta)$ and the properties of the mapping $\chi_\Delta(\zeta) \to \chi_\Delta(\mathcal{F}_E)$, which are particular cases of the properties formulated as Statements 1 - 4 of the Theorem 6.9. ☐

Theorem 6.11.

1. *If the set* Δ, $\Delta \subseteq \sigma_{\mathcal{F}_E}$*, is symmetric, that is* $\operatorname{mes} \Delta_a = 0$*, then the projector* $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ *is an orthogonal projector, i.e.*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = \mathcal{P}^*_{\mathcal{F}_E}(\Delta)\,. \tag{6.172}$$

2. *If the set* Δ, $\Delta \subseteq \sigma_{\mathcal{F}_E}$*, is not symmetric, i.e.* $\operatorname{mes} \Delta_a > 0$*, then the projector* $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ *is not orthogonal, i.e.* $\mathcal{P}_{\mathcal{F}_E}(\Delta) \neq \mathcal{P}^*_{\mathcal{F}_E}(\Delta)$.

Proof. Assume that the set Δ is symmetric. Then there are only two possibilities: either $\zeta_+(\mu) \in \Delta$, $\zeta_-(\mu) \in \Delta$, or $\zeta_+(\mu) \notin \Delta$, $\zeta_-(\mu) \notin \Delta$. According to (6.165), in the first case $\chi_\Delta(F(\mu)) = I$, in the second case $\chi_\Delta(F(\mu)) = 0$. In both cases, $\chi_\Delta(F(\mu)) = \chi^*_\Delta(F(\mu))$. By Definition 6.12,

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = M(\mathcal{L}), \quad \text{where} \quad M(\mu) = \chi_\Delta(F(\mu))\,.$$

Since $M(\mu) \equiv M^*(\mu)$, then according to Theorem 6.5, statement 4, $M(\mathcal{L}) = (M(\mathcal{L}))^*$.

If the set Δ is not symmetric, then, according to Lemma 6.12, $\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\| > 1$. Hence, the projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ is not orthogonal, but skew. $\square$

Theorem 6.12. *Let Δ_1 and Δ_2 are subsets of the spectrum $\sigma_{\mathcal{F}_E}$ which satisfy the condition*

$$\Big(\Delta_1 \cup (-\Delta_1)\Big) \bigcap \Big(\Delta_2 \cup (-\Delta_2)\Big) = \emptyset \,. \tag{6.173}$$

Then the image subspaces $\mathcal{P}_{\mathcal{F}_E}(\Delta_1)L^2(E)$ and $\mathcal{P}_{\mathcal{F}_E}(\Delta_2)L^2(E)$ are mutually orthogonal, that is

$$\mathcal{P}^*_{\mathcal{F}_E}(\Delta_2)\,\mathcal{P}_{\mathcal{F}_E}(\Delta_1) = 0 \,. \tag{6.174}$$

In particular, if

$$\text{either } \Delta_1 \subset \sigma^+_{\mathcal{F}_E},\ \Delta_2 \subset \sigma^+_{\mathcal{F}_E}, \quad\text{or}\quad \Delta_1 \subset \sigma^-_{\mathcal{F}_E},\ \Delta_2 \subset \sigma^-_{\mathcal{F}_E}, \tag{6.175a}$$

and moreover

$$\Delta_1 \cap \Delta_2 = \emptyset, \tag{6.175b}$$

then (6.174) *holds.*

Proof. The equality (6.174) is of the form $M_2^*(\mathcal{L}) \cdot M_1(\mathcal{L}) = 0$, where $M_1(\mu) = \chi_{\Delta_1}(F(\mu))$, $M_2(\mu) = \chi_{\Delta_2}(F(\mu))$. Using (6.165), we conclude that under the condition (6.173) at each point μ at least one of the matrices $M_1(\mu)$ and $M_2(\mu)$ vanishes. Thus, under the condition (6.173), the equality $M_2^*(\mu)M_1(\mu) \equiv 0$ holds. According to Theorem 6.5, the equality $M_2^*(\mathcal{L}) \cdot M_1(\mathcal{L}) = 0$ holds. $\square$

By induction with respect to n, from (6.170) the following statement can be derived:

Proposition 6.1. *Let Δ_k, $\Delta_k \subset \sigma_{\mathcal{F}_E}$, $1 \leq k \leq n$, be a finite sequence of sets possessing the properties:*

a). *Each of the sets Δ_k, $1 \leq k \leq n$, is $\mathcal{F}_E$-admissible;*

b). *The sets Δ_k, $1 \leq k \leq n$, are disjoint. This means that*

$$\Delta_p \cap \Delta_q = \emptyset, \quad \forall\, p,\, q:\ 1 \leq p,\, q \leq n,\, p \neq q \,.$$

Then the set $\Delta = \bigcup\limits_{1\leq k\leq n} \Delta_k$ *is admissible, and*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = \sum_{1\leq k\leq n} \mathcal{P}_{\mathcal{F}_E}(\Delta_k)\,. \tag{6.176}$$

The property of the mapping $\Delta \to \mathcal{P}_{\mathcal{F}_E}(\Delta)$ expressed as Proposition 6.1 can be naturally considered as *the additivity of this mapping with respect to* Δ.

In general, the mapping $\Delta \to \mathcal{P}_{\mathcal{F}_E}(\Delta)$ *is not countably additive.*If $\Delta_k,\ \Delta_k \subset \sigma_{\mathcal{F}_E},\ 1 \leq k < \infty,$ is a countable sequence of $\mathcal{F}_E$-admissible sets, then their union

$$\Delta = \bigcup_{1\leq k<\infty} \Delta_k \tag{6.177}$$

may be a non-admissible set. Even if the set Δ, (6.177), is admissible and the sets Δ_k are pairwise disjoint, the equality

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = \sum_{1\leq k<\infty} \mathcal{P}_{\mathcal{F}_E}(\Delta_k)\,.$$

may be violated. And what is more, it may happen that despite all the sets Δ_k, $1 \leq k < \infty$, and their union Δ are $\mathcal{F}_E$-admissible, the series in the right hand side of of the last formula may diverge in any reasonable sense and even $\|\mathcal{P}_{\mathcal{F}_E}(\Delta_k)\| \to \infty$ as $k \to \infty$. The fact that the union Δ of the countable sequence of $\mathcal{F}_E$-admissible sets Δ_k is a $\mathcal{F}_E$-admissible set does not forbid the following pathology: *the property of the sets* Δ_k *be essentially separated from zero may be not uniform with respect to* k. Each of the sets Δ_k may be fully asymmetric: $\Delta_k = (\Delta_k)_a$, but their union Δ may be symmetric: $\Delta = \Delta_s$, hence $\mathcal{F}_E$-admissible, Lemma 6.8. However if the property of the sets Δ_k be essentially separated from zero is *not uniform with respect to* k*,* then the sequence $\|\mathcal{P}_{\mathcal{F}_E}(\Delta_k)\|,\ 1 \leq k < \infty$, is unbounded.

Nevertheless, some restricted property of countable additivity of the mapping $\Delta \to \mathcal{P}_{\mathcal{F}_E}(\Delta)$ takes place.

Theorem 6.13. *Let* $\{\Delta_k\}_{1\leq k<\infty}$ *be a sequence of Borelian subsets of the spectrum* $\sigma_{\mathcal{F}_E}$ *possessing the following properties:*

1. *The sets* $\{\Delta_k\}_{1\leq k<\infty}$ *are pairwise disjoint:*

$$\Delta_p \cap \Delta_q = \emptyset \ \ \forall\ p,\, q:\ 1 \leq p,\, q < \infty,\, p \neq q\,. \tag{6.178}$$

2. *The sequence* $\{(\Delta_k)_a\}_{1\leq k<\infty}$ *of the asymmetric parts* $(\Delta_k)_a$ *of the sets* Δ_k *is uniformly essentially separated from zero, that is*

$$\inf_k \operatorname{ess\,dist}\bigl((\Delta_k)_a, 0\bigr) > 0\,. \tag{6.179}$$

Then the set $\Delta = \bigcup\limits_{1\leq k<\infty} \Delta_k$ *is* $\mathcal{F}_E$*-admissible and the equality*

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = \sum_{1\leq k<\infty} \mathcal{P}_{\mathcal{F}_E}(\Delta_k)\,. \tag{6.180}$$

holds, where the series in the right hand side of (6.180) *converges strongly.*

We preface the proof of Theorem 6.13 with an estimate of the matrix function $\chi_\Delta(F(\mu))$.

Lemma 6.10. *The norms of the spectral projectors* $E_+(\mu)$, $E_-(\mu)$ *of the matrix* $F(\mu)$ *are:*

$$\|E_+(\mu)\| = \cosh\frac{\pi\mu}{2}, \quad \|E_-(\mu)\| = \cosh\frac{\pi\mu}{2}\,, \quad 0<\mu<\infty\,. \tag{6.181a}$$

In terms of the eigenvalue $\zeta_\pm(\mu)$ *of the matrix* $F(\mu)$, (6.128a),

$$\begin{aligned} \|E_+(\mu)\| &= \frac{1}{2|\zeta_+(\mu)|}\sqrt{1+2|\zeta_+(\mu)|^2}, \\ \|E_-(\mu)\| &= \frac{1}{2|\zeta_-(\mu)|}\sqrt{1+2|\zeta_-(\mu)|^2}\,. \end{aligned} \tag{6.181b}$$

Proof. Calculation of the norm of the matrices $E_+(\mu)$, $E_-(\mu)$ can be reduced to calculation of the norms of the appropriate self-adjoint matrices:

$$\|E_+(\mu)\|^2 = \|E_+^*(\mu)\,E_+(\mu)\|, \quad \|E_-(\mu)\|^2 = \|E_-^*(\mu)\,E_-(\mu)\|$$

In its turn, calculation of the norm of a selfadjoin matrix can be reduced to calculation of its maximal eigenvalues. The characteristic equation for the matrix $\|E_+^*(\mu)\,E_+(\mu)\|$ is:

$$p^2 - (\operatorname{trace} E_+^*(\mu)\,E_+(\mu))p + \det E_+^*(\mu)\,E_+(\mu) = 0\,.$$

By direct computation, $\det E_+^*(\mu)\,E_+(\mu) = |\det E_+(\mu)|^2 = 0$, (recall that $E_+(\mu)$ is a matrix of rank one), and

$$\operatorname{trace} E_+^*(\mu)\,E_+(\mu) = \frac{1}{4}\biggl(2 + \frac{|f_{+-}(\mu)|^2}{|\zeta(\mu)|^2} + \frac{|f_{-+}(\mu)|^2}{|\zeta(\mu)|^2}\biggr)\,.$$

According to (6.58) and (6.139),

$$\frac{|f_{+-}(\mu)|^2}{|\zeta(\mu)|^2} = e^{\mu\pi}, \qquad \frac{|f_{-+}(\mu)|^2}{|\zeta(\mu)|^2} = e^{-\mu\pi}\,.$$

Thus,

$$\operatorname{trace} E_+^*(\mu)\, E_+(\mu) = \frac{1}{4}(2 + e^{\mu\pi} + e^{-\mu\pi}) = \cosh^2 \frac{\mu\pi}{2}\,.$$

So, the roots of the above characteristic equation are $p = 0$, and $p = \cosh^2 \frac{\mu\pi}{2}$. Therefore, $\|E_+^*(\mu)\, E_+(\mu)\| = \cosh^2 \frac{\mu\pi}{2}$, and $\|E_+(\mu)\| = \cosh \frac{\mu\pi}{2}$. Analogously, $\|E_-(\mu)\| = \cosh \frac{\mu\pi}{2}$. $\square$

Lemma 6.11. *Let the set* Δ, $\Delta \subseteq \sigma_{\mathcal{F}_E}$ *is non-empty:* $\operatorname{mes} \Delta > 0$.

1. *If the set* Δ *is symmetric, that is if* $\operatorname{mes} \Delta_a = 0$, *then*

$$\operatorname*{ess\,sup}_{\mu \in (0,\,\infty)} \|\chi_\Delta(F(\mu))\| = 1\,.$$

2. *If the set* Δ *is not symmetric, that is if* $\operatorname{mes} \Delta_a > 0$, *then*

$$\operatorname*{ess\,sup}_{\mu \in (0,\,\infty)} \|\chi_\Delta(F(\mu))\| = \frac{1}{2d}\sqrt{1 + 2d^2}\,,$$

where $d = \operatorname{ess\,dist}(0, \Delta_a)$. *In particular,*

$$\operatorname*{ess\,sup}_{\mu \in (0,\,\infty)} \|\chi_\Delta(F(\mu))\| > 1.$$

Proof. If $\Delta = \Delta_s$, then, according to (6.165), either $\chi_\Delta(F(\mu)) = I$, or $\chi_\Delta(F(\mu)) = 0$. Thus, in this case $\operatorname*{ess\,sup}_{\mu \in (0,\infty)} \|\chi_\Delta(F(\mu))\| = 1$. If $\operatorname{mes} \Delta_a > 0$, then according to (6.165) and (6.181), $\operatorname*{ess\,sup}_{\mu \in (0,\infty)} \|\chi_\Delta(F(\mu))\| = \frac{1}{2d}\sqrt{1 + 2d^2}$. In the considered case $d < \frac{1}{\sqrt{2}}$, thus $\frac{1}{2d}\sqrt{1 + 2d^2} < 1$. $\square$

Let us estimate norms of the operators $\mathcal{P}_{\mathcal{F}_E}(\Delta)$. In the case the function $h(\zeta) = \chi_\Delta(\zeta)$, the estimate (6.158) for $h(\mathcal{F}_E)$ can be improved.

Lemma 6.12. *Let the set* Δ, $\Delta \subseteq \sigma_{\mathcal{F}_E}$ *is non-empty:* $\operatorname{mes} \Delta > 0$.

1. *If the set* Δ *is symmetric, i.e.* $\operatorname{mes} \Delta_a = 0$, *then*

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\| = 1\,.$$

2. *If the set is not symmetric, i.e* $\operatorname{mes}\Delta_a > 0$*, then*

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\| = \frac{1}{2d}\sqrt{1+2d^2},$$

where $d = \operatorname{ess\,dist}(\Delta_a, 0)$ *. In particular,*

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\| > 1\,.$$

Proof. Lemma 6.12 is a consequence of Lemma 6.11 and Theorem 6.4. (Recall that $\mathcal{P}_{\mathcal{F}_E}(\Delta) = M(\mathcal{L})$, where $M(\mu) = \chi_\Delta(F(\mu))$.)

□

Proof of Theorem 6.13. Since

$$\Delta_a \subseteq \bigcup_{1\le k<\infty} (\Delta_k)_a\,,$$

then either $\operatorname{mes}(\Delta_a) = 0$ (in which case the set Δ is $\mathcal{F}_E$-admissible), or $\operatorname{mes}(\Delta_a) > 0$, in which case

$$\operatorname{ess\,dist}(0,\,\Delta_a) \ge \inf_k(\operatorname{ess\,dist}(0,\,(\Delta_k)_a)) > 0,$$

thus the set Δ is $\mathcal{F}_E$-admissible as well.

Thus, the set Δ, as well of the sets Δ_k, $1 \le k < \infty$, are $\mathcal{F}_E$-admissible. In particular, the operator $\mathcal{P}_{\mathcal{F}_E}(\Delta)$, as well as all the operators $\mathcal{P}_{\mathcal{F}_E}(\Delta_k)$, $1 \le k < \infty$, are defined.

From the assumption 1 of the present Theorem it follows that the $(n+1)$ sets, the sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ and the set $\bigcup\limits_{n<k<\infty} \Delta_k$, are disjoint, and

$$\Delta = \Big(\bigcup_{1\le k\le n} \Delta_k\Big) \bigcup \Big(\bigcup_{n<k<\infty} \Delta_k\Big)\,.$$

According to Proposition 6.1, the equality

$$\mathcal{P}_{\mathcal{F}_E}(\Delta) = \sum_{1\le k\le n} \mathcal{P}_{\mathcal{F}_E}(\Delta_k) \;+\; \mathcal{P}_{\mathcal{F}_E}\Big(\bigcup_{n<k<\infty} \Delta_k\Big)$$

holds. To prove (6.180), we have to prove that

$$\mathcal{P}_{\mathcal{F}_E}\Big(\bigcup_{n<k<\infty} \Delta_k\Big) \to 0 \quad \text{as} \quad n \to \infty\,,$$

where the convergence is the strong convergence. Decoding the notion of strong convergence, we present the limiting relation in the form

$$\|\mathcal{P}_{\mathcal{F}_E}\big(\bigcup_{n<k<\infty}\Delta_k\big)x\|_{L^2((0,\infty))} \to 0 \quad \text{as} \quad n\to\infty \quad \text{for every} \quad x\in L^2((0,\infty))\,. \tag{6.182}$$

Let us denote

$$u_n = \mathcal{P}_{\mathcal{F}_E}\big(\bigcup_{n<k<\infty}\Delta_k\big)x\,.$$

According to the definition of the operator $\mathcal{P}_{\mathcal{F}_E}\big(\bigcup\limits_{n<k<\infty}\Delta_k\big)$,

$$\widehat{u_n}(\mu) = \hat{x}(\mu)\mathcal{M}_n(\mu)\,, \tag{6.183}$$

where

$$\mathcal{M}_n(\mu) = \chi_{\bigcup\limits_{n<k<\infty}\Delta_k}\big(F(\mu)\big)\,. \tag{6.184}$$

According to Lemma 6.11 applied to the set $\bigcup\limits_{n<k<\infty}\Delta_k$, either $\|\mathcal{M}_n(\mu)\|\leq 1\ \ \forall\,\mu$ if this set is symmetric, or $\|\mathcal{M}_n(\mu)\|\leq \dfrac{1}{2\delta_n}\sqrt{1+2\delta_n^2}\ \ \forall\,\mu$, if this set is not symmetric, and $\delta_n = \operatorname{ess\,dist}\Big(0,\big(\bigcup\limits_{n<k<\infty}\Delta_k\big)_a\Big)$. However, $\delta_n \geq \inf\limits_{1\leq k<\infty} d_k$, where $d_k = \operatorname{ess\,dist}(0,\,(\Delta_k)_a)$. The numbers d_k are separated from zero: $d_k\geq d>0$. Therefore, $\|\mathcal{M}_n(\mu)\|\leq \dfrac{1}{2d}\sqrt{1+2d}\ \ \forall\,\mu$. So, in both cases

$$\|\mathcal{M}_n(\mu)\|\leq C<\infty, \tag{6.185}$$

where the value C does not depend on n and on μ.

From (6.183) and (6.185) it follows that

$$\widehat{u_n}(\mu)\widehat{u_n}^*(\mu) \leq C^2\,\widehat{x}(\mu)\widehat{x}^*(\mu)\,. \tag{6.186}$$

Since $\int\limits_{\mu\in(0,\infty)}\widehat{x}(\mu)\widehat{x}^*(\mu)\,\frac{d\mu}{2\pi}<\infty$, the inequality (6.186) means that the family $\{\widehat{u_n}(\mu)\widehat{u_n}^*(\mu)\}_n$ admits a summable majorant. From (6.183) and (6.184) it follows that

$$\lim_{n\to\infty}\widehat{u_n}(\mu)\widehat{u_n}^*(\mu) = 0 \qquad \text{for every} \quad \mu\in(0,\infty)\,. \tag{6.187}$$

From (6.187), (6.187) and Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty} \int\limits_{\mu\in(0,\infty)} \widehat{u_n}(\mu)\widehat{u_n}^*(\mu)\,\frac{d\mu}{2\pi} = 0\,,$$

hence $\|u_n\| \to 0$ as $n \to \infty$. The equality (6.182) is proved. Thereby the equality (6.180) is proved. $\square$

Definition 6.17. *To every $\mathcal{F}_E$-admissible set Δ , $\Delta \subseteq \sigma_{\mathcal{F}_E}$, we relate the subspace*

$$\mathcal{H}_{\mathcal{F}_E}(\Delta) \stackrel{\text{def}}{=} \mathcal{P}_{\mathcal{F}_E}(\Delta)L^2(\mathbb{R}_+)\,, \tag{6.188}$$

which is the image of the projector $\mathcal{P}_{\mathcal{F}_E}$.

Remark 6.11. *For every admissible Δ, the subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ is closed because it is the null-subspace of the bounded operator $\mathcal{I} \;-\; \mathcal{P}_{\mathcal{F}_E}(\Delta)$.*

Since $\mathcal{P}_{\mathcal{F}_E}(\emptyset) = 0$ and $\mathcal{P}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}) = \mathcal{I}$,

$$\mathcal{H}_{\mathcal{F}_E}(\emptyset) = 0, \quad \mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}) = L^2(\mathbb{R}_+)\,.$$

In view of (6.171), the subspace $\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus\ \Delta)$ is the null-space of the projector $\mathcal{P}_{\mathcal{F}_E}$:

$$\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus\ \Delta) = \{x \in L^2(\mathbb{R}_+) : \mathcal{P}_{\mathcal{F}_E}(\Delta)x = 0\} \tag{6.189}$$

and the subspaces $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ and $\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus \Delta)$ are complementary:

$$\mathcal{H}_{\mathcal{F}_E}(\Delta) \dotplus \mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus \Delta) = L^2(\mathbb{R}_+)\,. \tag{6.190}$$

(The sum in (6.190) is direct).

Since the projection operator $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ is a function of the operator $\mathcal{F}$, it commutes with $\mathcal{F}$:

$$\mathcal{P}_{\mathcal{F}_E}(\Delta)\,\mathcal{F} = \mathcal{F}\,\mathcal{P}_{\mathcal{F}_E}(\Delta)\,. \tag{6.191}$$

From (6.191) it follows that the pair of complementary subspaces $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ and $\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E} \setminus \Delta)$, (6.190), reduces the operator $\mathcal{F}_E$:

$$\mathcal{F}_E\mathcal{H}_{\mathcal{F}_E}(\Delta) \subseteq \mathcal{H}_{\mathcal{F}_E}(\Delta), \quad \mathcal{F}_E\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta) \subseteq \mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta)\,. \tag{6.192}$$

In particular, the subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ is invariant with respect to the operator $\mathcal{F}_E$, and one can consider the restriction $\mathcal{F}_E(\Delta)$ of the operator $\mathcal{F}_E$ onto its invariant subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$:

$$\mathcal{F}_E(\Delta) = \mathcal{F}_E\big|_{\mathcal{H}_{\mathcal{F}_E}(\Delta)}\,. \tag{6.193}$$

Theorem 6.14. *Let Δ be an admissible subset of the spectrum $\sigma_{\mathcal{F}_E}$ of the operator $\mathcal{F}_E$. The spectrum of the operator $\mathcal{F}_E(\Delta)$, which acts in the space $\mathcal{H}_{\mathcal{F}_E}(\Delta)$, is the essential closure* $\operatorname{essclos}(\Delta)$ *of the set Δ:*

$$\sigma_{\mathcal{F}_E(\Delta)} = \operatorname{ess\,clos}(\Delta)\,. \tag{6.194}$$

Proof.
1. Let $z \in \mathbb{C}$, $z \notin \operatorname{clos}(\Delta)$. Consider the function

$$f(\zeta) = \begin{cases} (\zeta - z)^{-1}\,, & \zeta \in \Delta; \\ \quad 0 \quad\ , & \zeta \notin \Delta. \end{cases} \tag{6.195}$$

The function $f(\zeta)$ is admissible: it is the product of two admissible functions, the function $\chi_\Delta(\zeta)$ and the function $g(\zeta) = (\zeta - z)^{-1}$, $\forall\zeta$. First of these functions is admissible because the set Δ is admissible, second of these functions is admissible because it is essentially bounded on $\sigma_{\mathcal{F}_E}$ and smooth at the point $\zeta = 0$. Since $f(\zeta)(\zeta - z) = \chi_\Delta(\zeta,$

$$(\mathcal{F}_E - z\mathfrak{I})f(\mathcal{F}_E) = f(\mathcal{F}_E)(\mathcal{F}_E - z\mathfrak{I}) = \mathcal{P}_{\mathcal{F}_E}(\Delta)\,.$$

If we restrict this equality on the subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$, which is an invariant subspace for both the operators $\mathcal{F}_E$ and $f(\mathcal{F}_E)$, we come to the equality

$$\begin{aligned} \big(\mathcal{F}_E(\Delta) - z\mathfrak{I}(\Delta)\big)f(\mathcal{F}_E)_{|_{\mathcal{H}_{\mathcal{F}_E}(\Delta)}} &= \\ &= f(\mathcal{F}_E)_{|_{\mathcal{H}_{\mathcal{F}_E}(\Delta)}}\big(\mathcal{F}_E(\Delta) - z\mathfrak{I}(\Delta)\big) = \mathfrak{I}(\Delta), \end{aligned}$$

where $\mathfrak{I}(\Delta)$ is the identical operator in the space $\mathcal{H}_{\mathcal{F}_E}(\Delta)$. Thus, the operator $\mathcal{F}_E(\Delta) - z\mathfrak{I}(\Delta)$ is invertible operator acting in the space $\mathcal{H}_{\mathcal{F}_E}(\Delta)$.
2. Let $z \in \operatorname{essclos}(\Delta)$. This means that

$$\operatorname*{ess\,inf}_{\zeta\in\Delta} |\zeta - z| = 0\,.$$

We construct a sequence $\{x_n\}_{1\leq n<\infty}$ such that $x_n \in \mathcal{H}_{\mathcal{F}_E}(\Delta)$, $\|x_n\| = 1$, but $\|\big(\mathcal{F}_E(\Delta) - z\mathfrak{I}(\Delta)\big)x_n\| \to 0$ as $n \to \infty$. This construction is similar to the construction used by the proof of the second part of the statement 5 of Theorem 6.9. $\square$

Theorem 6.14 justifies the following

Definition 6.18. *Let Δ, $\Delta \in \sigma_{\mathcal{F}_E}$, be a $\mathcal{F}_E$-admissible set. The projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ defined by* (6.164) *is said to be the $\mathcal{F}_E$* spectral projector corresponding to the set Δ.

The subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ – the image of the operator $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ – is said to be the $\mathcal{F}_E$ spectral subspace corresponding to the set Δ.

For $0 < \varepsilon \leq 1/\sqrt{2}$, let

$$\Delta_+(\varepsilon) = e^{i\pi/4}\big[\varepsilon,\, 1/\sqrt{2}\big], \quad \Delta_-(\varepsilon) = e^{i\pi/4}\big[-1/\sqrt{2}, -\varepsilon\big]\,. \tag{6.196}$$

Each of the sets $\Delta_+(\varepsilon), \Delta_-(\varepsilon)$ with $\varepsilon > 0$ is $\mathcal{F}_E$-admissible, however the norms of spectral projectors $\mathcal{P}_{\mathcal{F}_E}(\Delta_+(\varepsilon))$, $\mathcal{P}_{\mathcal{F}_E}(\Delta_-(\varepsilon))$ tend to ∞ as $\varepsilon \to +0$. Indeed, the sets $\Delta_\pm(\varepsilon))$ are fully asymmetric:

$$\Delta_+(\varepsilon) = (\Delta_+(\varepsilon))_a, \quad \Delta_-(\varepsilon) = (\Delta_+(\varepsilon))_a.$$

It is clear that

$$\operatorname{ess\,dist}\big(\Delta_+(\varepsilon))_a, 0\big) = \varepsilon, \quad \operatorname{ess\,dist}\big(\Delta_-(\varepsilon))_a, 0\big) = \varepsilon$$

According to Lemma 6.12,

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta_+(\varepsilon))\| = \frac{1}{2\varepsilon}\,\sqrt{1+2\varepsilon^2}, \quad \|\mathcal{P}_{\mathcal{F}_E}(\Delta_-(\varepsilon))\| = \frac{1}{2\varepsilon}\,\sqrt{1+2\varepsilon^2}\,. \tag{6.197}$$

In particular,

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta_+(\varepsilon))\| \to +\infty, \quad \|\mathcal{P}_{\mathcal{F}_E}(\Delta_-(\varepsilon))\| \to +\infty \text{ as } \varepsilon \to +0\,. \tag{6.198}$$

At the same time, the set

$$\Delta(\varepsilon) = \Delta_+(\varepsilon) \cup \Delta_-(\varepsilon) \tag{6.199}$$

is symmetric: $\big(\Delta(\varepsilon)\big)_a = \emptyset$. According to Lemma 6.12,

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon))\| = 1 \quad \text{for every} \quad \varepsilon > 0,$$

or

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta_+(\varepsilon)) + \mathcal{P}_{\mathcal{F}_E}(\Delta_-(\varepsilon))\| = 1 \quad \text{for every} \quad \varepsilon > 0. \tag{6.200}$$

Sums of projectors from two unbounded families form a bounded family.

The family $\{\Delta(\varepsilon)\}_{\varepsilon>0}$ is monotonic: $\Delta(\varepsilon_1) \subseteq \Delta(\varepsilon_1)$ if $\varepsilon_1 > \varepsilon_2$. Moreover, $\bigcup\limits_{\varepsilon>0} \Delta(\varepsilon) = \sigma_{\mathcal{F}_E}\backslash 0$. Since $\mathcal{P}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\backslash 0) = \mathcal{I}$, then, according to Theorem 6.13, the following assertion holds:

Lemma 6.13. *The estimate*

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon))\| = 1 \quad \text{for every} \quad \varepsilon > 0. \tag{6.201}$$

and the limiting relation

$$\lim_{\varepsilon \to +0} \mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon)) = \mathcal{I}\,, \tag{6.202}$$

hold, where convergence is the strong convergence of operators.

Corollary 6.3. *As we saw,*

$$\sup_{\Delta} \|\mathcal{P}_{\mathcal{F}_E}(\Delta)\| = \infty\,, \tag{6.203}$$

where Δ runs over the class of all $\mathcal{F}_E$-admissible sets. From this it follows that the family $\{\mathcal{P}_{\mathcal{F}_E}(\Delta)\}$ of spectral projectors is not similar to an orthogonal family of projectors.

Lemma 6.14. *By contrast with* (6.203)*,*

$$\|\mathcal{F}_E \mathcal{P}_{\mathcal{F}_E}(\Delta)\| \le 1 \;\; \text{for every admissible} \;\; \Delta\,. \tag{6.204}$$

Proof. By definition of function of matrix,

$$F(\mu)\chi_\Delta\big(F(\mu)\big) = F(\mu)\Big(\chi_\Delta(\zeta(\mu))E_1(\mu) + \chi_\Delta(-\zeta(\mu))E_2(\mu)\Big)\,.$$

Explicit calculation gives:

$$F(\mu)E_+(\mu) = \begin{bmatrix} \frac{\zeta(\mu)}{2} & \frac{f_{-+}(\mu)}{2} \\ \frac{f_{+-}(\mu)}{2} & \frac{\zeta(\mu)}{2} \end{bmatrix}, \quad F(\mu)E_-(\mu) = \begin{bmatrix} -\frac{\zeta(\mu)}{2} & \frac{f_{-+}(\mu)}{2} \\ \frac{f_{+-}(\mu)}{2} & -\frac{\zeta(\mu)}{2} \end{bmatrix}.$$

Since

$$\frac{|\zeta(\mu)|}{2} + \frac{|f_{-+}(\mu)|}{2} < 1, \quad \frac{|\zeta(\mu)|}{2} + \frac{|f_{+-}(\mu)|}{2} < 1\,,$$

then

$$\|F(\mu)E_+(\mu)\| \le 1, \quad \|F(\mu)E_-(\mu)\| \le 1\,.$$

Since

$$F(\mu)\chi_\Delta\big(F(\mu)\big) = \chi_\Delta(\zeta_+(\mu))\,F(\mu)E_+(\mu) + \chi_\Delta(\zeta_-(\mu))\,F(\mu)E_-(\mu)$$

and $E_+(\mu) + E_-(\mu) = 1$, $\|F(\mu)\| \le 1$, then

$$\|F(\mu)\chi_\Delta\big(F(\mu)\big)\| \le 1, \quad 0 < \mu < \infty\,.$$

Finally $\|\mathcal{F}_E\chi_\Delta(\mathcal{F}_E)\| \le 1$, i.e. the estimate (6.204) holds. $\square$

In fact, we have proved the following estimate.

Lemma 6.15. *Let $f(\zeta)$ be any $\mathcal{F}_E$-admissible function. Then*

$$\|\mathcal{F}_E f(\mathcal{F}_E)\| \le 2 \operatorname*{ess\,sup}_{\zeta \in \sigma_{\mathcal{F}_E}} |f(\zeta)| \,. \tag{6.205}$$

7. After the spectral projectors $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ were introduced, (6.164), and investigated, (see in particular Lemma (6.14)), the question arises: how to represent the original operator $\mathcal{F}_E$ in terms of these spectral projectors. Our goal here is to give a meaning to the representation

$$\mathcal{F}_E = \int\limits_{\sigma_{\mathcal{F}_E}} \zeta \, \mathcal{P}_{\mathcal{F}_E}(d\zeta) \,, \tag{6.206}$$

and more generally,

$$f(\mathcal{F}_E) = \int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta) \, \mathcal{P}_{\mathcal{F}_E}(d\zeta) \,. \tag{6.207}$$

We emphasize that the operator $\mathcal{F}_E$ is non-normal, the family $\{\mathcal{P}_{\mathcal{F}_E}(\Delta)\}$ is not orthogonal and even unbounded: (6.203). However it turns out that if the interval Δ is essentially separated from zero:

$$\operatorname{ess\,dist}(\Delta, 0) > 0, \tag{6.208}$$

and the function $f(\zeta)$ is bounded on Δ, then the integral $\int\limits_{\Delta} f(\zeta) \, \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ can be provided with a meaning.

Namely, let $g(\zeta)$ be a simple function, that is the function of the form

$$g(\zeta) = \sum_k a_k \chi_{\Delta_k}(\zeta) \,, \tag{6.209}$$

where a_k are come complex numbers, and the collections Δ_k of sets forms a partition (finite) of the original set Δ: $\Delta = \bigcup\limits_k \Delta_k, \;\; \Delta_p \cap \Delta_q = \emptyset, \; p \ne q$. We define the integral $\int\limits_{\Delta} g(\zeta) \; \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ as

$$\int\limits_{\Delta} g(\zeta) \; \mathcal{P}_{\mathcal{F}_E}(d\zeta) \stackrel{\text{def}}{=} \sum_k a_k \mathcal{P}_{\mathcal{F}_E}(\Delta_k) \,. \tag{6.210}$$

The value of the sum in the right hand side of (6.210) does not depend on the representation of the function g in the form (6.209). So, the

value in the left hand side of (6.210) is well defined. From the other hand, decoding definition of $\mathcal{P}_{\mathcal{F}_E}(\Delta_k)$ as $\chi_{\Delta_k}(\mathcal{F}_E)$, we have

$$\sum_k a_k \mathcal{P}_{\mathcal{F}_E}(\Delta_k) = \sum_k a_k \chi_{\Delta_k}(\mathcal{F}_E) = \big(\sum_k a_k \chi_{\Delta_k}\big)(\mathcal{F}_E) = g(\mathcal{F}_E)\,,$$

and finally,

$$\int\limits_{\Delta} g(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta) = g(\mathcal{F}_E)\,. \tag{6.211}$$

So *for any simple function $g(\zeta)$ vanishing outside the set Δ, where Δ is separated from zero, the integral $\int\limits_{\Delta} g(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ is well defined and is interpreted as a function of the operator $\mathcal{F}_E$ in the sense of the above introduced functional calculus.*

Given a function f bounded on Δ and vanishing outside of Δ, there exists sequence f_n of simple functions vanishing outside of Δ which converges to f uniformly on Δ:

$$\overline{\lim_{n\to\infty}} \sup_{\zeta\in\Delta} \big|f(\zeta) - f_n(\zeta)\big| = 0\,.$$

The integral $\int\limits_{\Delta} f(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ will be defined as the limit of integrals $\int\limits_{\Delta} f_n(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ of simple functions f_n if we justify that such a limit exists and does not depend on the approximating sequence $\{f_n\}$.

If $h(\zeta)$ be a function essentially bounded on Δ and vanishing outside of Δ, then

$$\|h(\mathcal{F})\| \leq \frac{2}{d} \sup_{\zeta\in\Delta} |h(\zeta)|,$$

where $d = \operatorname{ess\,dist}(\Delta, 0)$. (See (6.137)). Applying this estimate to $h = f - f_n$, we see that $\|f(\mathcal{F}) - f_n(\mathcal{F})\| \to 0$ as $n \to \infty$. According to (6.211), this can be presented as

$$\Big\| f(\mathcal{F}_E) - \int\limits_{\Delta} f_n(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta) \Big\| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Thus, there exists the limit of integrals $\int\limits_{\Delta} f_n(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta)$. We declear this limit as $\int\limits_{\Delta} f(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta)$:

$$\int\limits_{\Delta} f(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta) \stackrel{\text{def}}{=} \lim_{n\to\infty} \int\limits_{\Delta} f_n(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta),$$

where convergence is the convergence in the norm of operators acting in $L^2(\mathbb{R}_+)$.

So *the integral* $\int\limits_{\Delta} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ *is defined if* Δ *is any subset of* $\sigma_{\mathcal{F}_E}$ *separated from zero and* f *is any function bounded on* Δ *and vanishing outside* Δ. Moreover, this integral can be interpreted as a function f of the operator $\mathcal{F}_E$ in the sense of Definition 6.12:

$$\int\limits_{\Delta} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta) = f(\mathcal{F}_E).$$

Let now f be any bounded function defined on the spectrum $\sigma_{\mathcal{F}_E}$. (We emphasize that the spectrum $\sigma_{\mathcal{F}_E}$ is not separated from zero, but contains the zero point, which is the singular point in some sense: see (6.198).) The integral $\int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ will be defined as an unproper integral. We remove a *symmetric* ε - neighborhood V_ε of zero

$$V_\varepsilon = \big(-\varepsilon e^{i\pi/4},\, \varepsilon e^{i\pi/4}\big), \tag{6.212}$$

from the spectrum $\sigma_{\mathcal{F}_E}$ and integrate f over the set $\Delta(\varepsilon) = \sigma_{\mathcal{F}_E} \setminus V_\epsilon$. (This is the same set $\Delta(\varepsilon)$ that was already defined in (6.196), (6.199).) The set $\Delta(\varepsilon)$ is separated from zero, so the integral $\int\limits_{\Delta(\varepsilon)} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)$ is already defined. Then we pass to the limit as $\varepsilon \to +0$. If the limits exists in some sense, we decare the limiting operator as the integral $\int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)$:

$$\int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)x \stackrel{\text{def}}{=} \lim_{\varepsilon\to+0} \int\limits_{\sigma_{\mathcal{F}_E}\setminus V_\varepsilon} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)\,. \tag{6.213}$$

Lemma 6.16. *We assume that* f *is a* $\mathcal{F}_E$*-admissible function.*

Then the limit in (6.213) *exists in the sense of strong convergence, that is for every* $x \in L^2(\mathbb{R}_+)$,

$$\Big\| \int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)x - \int\limits_{\sigma_{\mathcal{F}_E}\setminus V_\varepsilon} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta)x \Big\|_{L^2(\mathbb{R}_+)} \to 0 \;\text{ as }\; \varepsilon \to +0\,. \tag{6.214}$$

Moreover

$$\int\limits_{\sigma_{\mathcal{F}_E}} f(\zeta)\; \mathcal{P}_{\mathcal{F}_E}(d\zeta) = f(\mathcal{F}_E)\,, \tag{6.215}$$

where the operator $f(\mathcal{F}_E)$ is defined in the sense the above developed functional calculus for the operator $\mathcal{F}_E$, (Definition 6.12).

Proof. To justify the limiting relation (6.214) and to establish the equality (6.215), we observe that

$$\int\limits_{\sigma_{\mathcal{F}_E}\setminus V_\varepsilon} f(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta) = \int\limits_{\Delta(\varepsilon)} \chi_{\Delta(\varepsilon)}(\zeta) f(\zeta) \mathcal{P}_{\mathcal{F}_E}(d\zeta) = \big(\chi_{\Delta(\varepsilon)} f\big)(\mathcal{F}_E).$$

(For functions vanishing outside the set $\Delta(\varepsilon)$, which is separated from zero, the equality (6.215) is already established. In the present case, we apply the equality (6.215) to the function $\chi_{\Delta(\varepsilon)}(\zeta)f(\zeta)$.) Since

$$\big(\chi_{\Delta(\varepsilon)} f\big)(\mathcal{F}_E) = \chi_{\Delta(\varepsilon)}(\mathcal{F}_E) f(\mathcal{F}_E) = \mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon)) f(\mathcal{F}_E)\,,$$

we have

$$\int\limits_{\sigma_{\mathcal{F}_E}\setminus V_\varepsilon} f(\zeta)\ \mathcal{P}_{\mathcal{F}_E}(d\zeta) = \mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon))\, f(\mathcal{F}_E)\,.$$

According to Lemma 6.13,

$$\lim_{\varepsilon\to+0} \mathcal{P}_{\mathcal{F}_E}(\Delta(\varepsilon))\, f(\mathcal{F}_E) = f(\mathcal{F}_E)\,,$$

where convergence is the strong convergence of operators. Thus under the assumptions of Lemma, there exists the strong limit in (6.213) and the equality (6.215) holds. ☐

7. Let us find the generalized eigenfunction of the operator $\mathcal{F}_E$. According to the enumeration agreement, (see Remark 6.9), each points $\zeta \in \sigma_{\mathcal{F}_E}\setminus 0$ is uniquely representable as $\zeta = \zeta_s(\mu)$, with $\mu \in [0,\infty)$, $s \in \{+,\,-\}$, where $\zeta_s(\mu)$ were defined in (6.128). We extend this enumeration agreement, enumerating the generalized eigenfunction of the operator $\mathcal{F}_E$ which correspond to the point $\zeta_s(\mu) \in \sigma_{\mathcal{F}_E} \setminus 0$ by the same double index $(\mu,\, s)$. (To each $\zeta \in \sigma_{\mathcal{F}_E} \setminus 0$ corresponds only one, up to proportionality, generalized eigenfunction.) Thus, the eigenfunction of $\mathcal{F}_E$ which corresponds to the eigenvalue $\zeta_s(\mu) \in \sigma_{\mathcal{F}_E} \setminus 0$ is denoted by $u_s(\,.\,,\,\mu)$:

$$\mathcal{F}_E u_s(\,.\,,\,\mu) = \zeta_s(\mu)\, u_s(\,.\,,\,\mu).$$

The expressions for the eigenfunctions $u_s(\,.\,,\,\mu)$ have a little bit different forms depending on to which part either $\sigma^+_{\mathcal{F}_E}$ or $\sigma^-_{\mathcal{F}_E}$ of the spectrum $\sigma_{\mathcal{F}_E}$ belongs the eigenvalue $\zeta_s(\mu)$, that is on the sign s.

Let $\Delta \subset \sigma^{+}_{\mathcal{F}_E}$ be an $\mathcal{F}_E$-admissible set, and $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ be a corresponding spectral projector. According to Definitions 6.12 and 6.7, the projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ is an integral operator:

$$(\mathcal{P}_{\mathcal{F}_E}(\Delta)x)(t) = \int\limits_{\mathbb{R}_+} x(\omega) K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(\omega, t)\, d\omega,$$

where the kernel of this operator is of the form

$$K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t, \omega) = \int\limits_{\mu \in \zeta_{+}^{[-1]}(\Delta)} k_{+}(\omega, t; \mu) \frac{d\mu}{2\pi}\,. \tag{6.216}$$

where

$$k_{+}(\omega, t; \mu) = \psi^{*}(\omega, \mu) E_{+}(\mu) \psi(t, \mu)\,. \tag{6.217}$$

Here $\zeta_{+}^{[-1]} : \sigma^{+}_{\mathcal{F}_E} \to \mathbb{R}_+$ is a mapping inverse to the mapping $\zeta_{+} : \mathbb{R}_+ \to \sigma^{+}_{\mathcal{F}_E}$, (6.128a).

Analogously, if $\Delta \subset \sigma^{-}_{\mathcal{F}_E}$ be an $\mathcal{F}_E$-admissible set, and $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ be a corresponding spectral projector, then the kernel $K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t, \omega)$ of the spectral projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ is of the form

$$K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t, \omega) = \int\limits_{\mu \in \zeta_{-}^{[-1]}(\Delta)} k_{-}(\omega, t; \mu) \frac{d\mu}{2\pi}\,, \tag{6.218}$$

where

$$k_{-}(\omega, t; \mu) = \psi^{*}(\omega, \mu) E_{-}(\mu) \psi(t, \mu)\,. \tag{6.219}$$

Here $\zeta_{-}^{[-1]} : \sigma^{-}_{\mathcal{F}_E} \to \mathbb{R}_+$ is a mapping inverse to the mapping $\zeta_{-} : \mathbb{R}_+ \to \sigma^{-}_{\mathcal{F}_E}$, (6.128a).

In general case, in which Δ may not be contained neither in $\sigma^{+}_{\mathcal{F}_E}$, not in $\sigma^{-}_{\mathcal{F}_E}$, the kernel $K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t, \omega)$ of the spectral projector is of the form

$$K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t, \omega) = \int\limits_{\mu \in \zeta_{+}^{[-1]}(\Delta^{+})} k_{+}(\omega, t; \mu) \frac{d\mu}{2\pi} + \int\limits_{\mu \in \zeta_{-}^{[-1]}(\Delta^{-})} k_{-}(\omega, t; \mu) \frac{d\mu}{2\pi}\,, \tag{6.220a}$$

where

$$\Delta^{+} = \Delta \cap \sigma^{+}_{\mathcal{F}_E}, \quad \Delta^{-} = \Delta \cap \sigma^{-}_{\mathcal{F}_E}\,. \tag{6.220b}$$

The representation (6.220) can be presented in the "unified" form

$$K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t,\omega) = \sum_{s=+,-} \int\limits_{\mu\in\zeta_s^{[-1]}(\Delta^s)} k_s(\omega,t;\mu)\frac{d\mu}{2\pi}\,. \tag{6.221}$$

The matrix $E_+(\mu)$, which is of rank one, admits the factorization

$$E_+(\mu) = \frac{1}{2}\begin{bmatrix} 1 \\ \dfrac{f_{+-}(\mu)}{\zeta_+(\mu)} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix}\,. \tag{6.222}$$

(We take into account the equality (6.139): $\zeta_+^2(\mu) = f_{-+}(\mu)f_{+-}(\mu)$). Thus the "elementary kernel" $k_+(\omega,t;\mu)$ is representable as the product

$$k_+(\omega,t;\mu) = \overline{v_+(\omega,\mu)}\; u_+(t,\mu)\,, \tag{6.223}$$

where for $0\leq\mu<\infty$

$$u_+(t,\mu) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix}\begin{bmatrix} \psi_+(t,\,\mu) \\ \psi_-(t,\,\mu) \end{bmatrix}\,, \tag{6.224}$$

and

$$v_+(t,\mu) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \dfrac{\overline{f_{+-}(\mu)}}{\overline{\zeta_+(\mu)}} \end{bmatrix}\begin{bmatrix} \psi_+(t,\,\mu) \\ \psi_-(t,\,\mu) \end{bmatrix}\,. \tag{6.225}$$

The matrix $E_-(\mu)$, which is of rank one, admits the factorization

$$E_-(\mu) = \frac{1}{2}\begin{bmatrix} 1 \\ \dfrac{f_{+-}(\mu)}{\zeta_-(\mu)} \end{bmatrix} \begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_-(\mu)} \end{bmatrix}\,. \tag{6.226}$$

(Now we use the equality (6.139): $\zeta_-^2(\mu) = f_{-+}(\mu)f_{+-}(\mu)$). Thus the 'elementary kernel" $k_-(\omega,t;\mu)$ is representable as the product

$$k_-(\omega,t;\mu) = \overline{v_-(\omega,\mu)}\; u_-(t,\mu)\,, \tag{6.227}$$

where for $0<\mu<\infty$

$$u_-(t,\mu) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_-(\mu)} \end{bmatrix}\begin{bmatrix} \psi_+(t,\,\mu) \\ \psi_-(t,\,\mu) \end{bmatrix}\,, \tag{6.228}$$

and

$$v_-(t,\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \overline{\frac{f_{+-}(\mu)}{\zeta_-(\mu)}} \end{bmatrix} \begin{bmatrix} \psi_+(t,\,\mu) \\ \psi_-(t,\,\mu) \end{bmatrix} . \tag{6.229}$$

The general ideology, [Pov1], [Pov2], [Mau], says. *If the kernel $K_{\mathcal{P}_{\mathcal{F}_E}(\Delta)}(t,\omega)$ of the spectral projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$ is representable as the integral of the elementary kernels $k_\pm(\omega,t;\mu)$ of the form* (6.223)-(6.227)*, then the functions $u_+(t,\mu)$, $u_-(t,\mu)$ are the generalized eigenfunction of the operator $\mathcal{F}_E$, and the function $v_+(t,\mu)$, $v_-(t,\mu)$ are the generalized eigenfunction of the operator $\mathcal{F}_E^*$.*

The fact that the function $u_+(t,\mu)$, $u_-(t,\mu)$ are eigenfunctions of the operator $\mathcal{F}_E$:

$$\mathcal{F}_E u_+(\,.\,,\mu) = \zeta_+(\mu) u_+(\,.\,,\mu)\,, \tag{6.230a}$$

$$\mathcal{F}_E u_-(\,.\,,\mu) = \zeta_-(\mu) u_-(\,.\,,\mu)\,, \tag{6.230b}$$

can be confirmed by the direct calculation. Performing this direct calculation, we first of all interpret the operator $\mathcal{F}_E$ as an integral operator:

$$(\mathcal{F}_E x)(t) = \int\limits_{\omega\in(0,\infty)} x(\omega) K_{\mathcal{F}_E}(\omega,t)\, d\omega\,, \tag{6.231}$$

where the kernel $K_{\mathcal{F}_E}(\omega,t)$ is of the form

$$K_{\mathcal{F}_E}(\omega,t) = \int\limits_{\mu\in(0,\infty)} \psi^*(\omega,\mu)) F(\mu) \psi(t,\mu) \frac{d\omega}{2\pi}\,. \tag{6.232}$$

For example, to obtain (6.230a), we substitute the expression

$$u_+(\omega,\mu) = \begin{bmatrix} 1 & \frac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix} \psi(\omega,\mu)$$

as $x(\omega)$ into (6.231), where $K_{\mathcal{F}_E}(\omega,t)$ is represented by the integral (6.232). Then we change the order of integration. We get

$$(\mathcal{F}_E u_+(\,.\,,\mu))(t) = \\ \int\limits_{\mu\in(0,\infty)} \Bigg(\int\limits_{\omega\in(0,\infty)} u_+(\omega,\mu)\psi^*(\omega,\mu')) \frac{d\omega}{2\pi} \Bigg) F(\mu')\psi(t,\mu')\, d\mu'\,.$$

Then we use the identity (6.36):

$$\int\limits_{(0,\infty)} \psi(\omega,\mu)\psi^*(\omega,\mu')\,d\omega = 2\pi\,\delta(\mu-\mu')I$$

and the fact that vector-row $\begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix}$ is an eigenvector of the matrix $F(\mu)$:

$$\begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix} F(\mu) = \zeta_+(\mu) \begin{bmatrix} 1 & \dfrac{f_{-+}(\mu)}{\zeta_+(\mu)} \end{bmatrix}.$$

In the same way that we obtained the equalities (6.230) we can obtain the equalities

$$\mathcal{F}_E^* v_+(\,.\,,\mu) = \overline{\zeta_+}(\mu) v_+(\,.\,,\mu)\,, \tag{6.233a}$$

$$\mathcal{F}_E^* v_-(\,.\,,\mu) = \overline{\zeta_-}(\mu) v_-(\,.\,,\mu)\,. \tag{6.233b}$$

Remark 6.12. *The equalities* (6.230), (6.233) *means that the functions $u_+(t,\mu)$, $u_-(t,\mu)$ are eigenfunctions of the operator $\mathcal{F}_E$, and the functions $v_+(t,\mu)$, $v_-(t,\mu)$ are eigenfunctions of the operator $\mathcal{F}_E^*$. These eigenfunctions are generalized: they do not belong to $L^2(\mathbb{R}_+)$. Deriving the equalities* (6.230), (6.233)*, we performed formal manipulations. These manipulations can be justified using the regularization procedure.*

Remark 6.13. *For $\mu > 0$, the eigenfunctions $u_+(t,\mu)$ and $u_-(t,\mu)$, corresponding to the eigenvalues $\zeta_+(\mu)$ and $\zeta_-(\mu)$ respectively:*

$$u_+(t,\mu) = 2^{-1/2}\,t^{-1/2}\left(t^{i\mu} + \frac{f_{-+}(\mu)}{\zeta(\mu)}t^{-i\mu}\right), \tag{6.234}$$

$$u_-(t,\mu) = 2^{-1/2}\,t^{-1/2}\left(t^{i\mu} - \frac{f_{-+}(\mu)}{\zeta(\mu)}t^{-i\mu}\right). \tag{6.235}$$

oscillate.

The eigenfunction $u_+(t,0)$, corresponding to the eigenvalue $\zeta_+(0) = \frac{1}{\sqrt{2}}e^{i\pi/4}$, is

$$u_+(t,0) = 2^{1/2}\,t^{-1/2}\,. \tag{6.236}$$

This function do not oscillate.

The function $u_-(t,0)$ *vanishes identically:* $u_-(t,0)\equiv 0$. *The eigenfunction corresponding to the eigenvalue* $\zeta_-(0) = -\frac{1}{\sqrt{2}}e^{i\pi/4}$, *can be obtained as*

$$\frac{d}{d\mu}u_-(t,\mu)_{|\mu=0} = 2^{1/2}it^{-1/2}\left(\ln t - \frac{1}{2\sqrt{\pi}}\Gamma'(1/2) - i\frac{\pi}{4}\right). \quad (6.237)$$

This function also do not oscillate.

8. Let us consider the orthogonality properties of the generalized eigenfunction of the operator $\mathcal{F}_E$. This consideration is based on the equality (6.36).

Let $0 \le \mu'$, $\mu'' < \infty$. From (6.224) and (6.36) it follows that

$$\int\limits_{t\in(0,\infty)} u_+(t,\mu')\,\overline{u_+(t,\mu'')}\,dt =$$

$$= \frac{1}{2}\begin{bmatrix}1 & \dfrac{f_{-+}(\mu')}{\zeta_+(\mu')}\end{bmatrix}\begin{bmatrix}1 & \dfrac{f_{-+}(\mu'')}{\zeta_+(\mu'')}\end{bmatrix}^{*}\delta(\mu'-\mu'')$$

Taking into account the equalities (6.139) and (6.58), we obtain that

$$\int\limits_{t\in(0,\infty)} u_+(t,\mu')\,\overline{u_+(t,\mu'')}\,dt = \frac{1}{2}\left(1+e^{\mu'\pi}\right)\delta(\mu'-\mu'')\,.$$

In the same way,

$$\int\limits_{t\in(0,\infty)} u_-(t,\mu')\,\overline{u_-(t,\mu'')}\,dt = \frac{1}{2}\left(1+e^{\mu'\pi}\right)\delta(\mu'-\mu'')\,.$$

and

$$\int\limits_{t\in(0,\infty)} u_+(t,\mu')\,\overline{u_-(t,\mu'')}\,dt = \frac{1}{2}\left(1-e^{\mu'\pi}\right)\delta(\mu'-\mu'')\,.$$

These orthogonality relations for the eigenfunctions $u_s(t,\mu)$ of the operator $\mathcal{F}_E$ can be uniformly presented as

$$\int\limits_{t\in(0,\infty)} u_{s'}(t,\mu')\,\overline{u_{s''}(t,\mu'')}\,dt =$$

$$= \frac{1}{2}\left(1+(-1)^{s'+s''}e^{(\mu'+\mu'')/2\pi}\right)\delta(\mu'-\mu'')\,,$$

$$\forall\ \mu',\ \mu''\in(0,\infty),\quad \forall\ s',\ s''\in\{+,-\}\,. \quad (6.238)$$

Analogously, orthogonality relations for the eigenfunctions $v_s(t,\mu)$ of the adjoint operator $\mathcal{F}_E^*$ can be uniformly presented as

$$\int\limits_{t\in(0,\infty)} v_{s'}(t,\mu')\,\overline{v_{s''}(t,\mu'')}\,dt =$$
$$= \frac{1}{2}\left(1 + (-1)^{s'+s''} e^{-(\mu'+\mu'')/2\pi}\right)\delta(\mu'-\mu'')\,,$$
$$\forall\ \ \mu'\,,\,\mu''\in(0,\infty),\ \ \forall\ \ s',\,s''\in\{+,-\}\,. \quad (6.239)$$

Finally, the biorthogonality relations between the the eigenfunctions $u_s(t,\mu)$ of the operator $\mathcal{F}_E$ and the eigenfunctions $v_s(t,\mu)$ of the adjoint operator $\mathcal{F}_E^*$ are:

$$\int\limits_{t\in(0,\infty)} u_{s'}(t,\mu')\,\overline{v_{s''}(t,\mu'')}\,dt =$$
$$= \frac{1}{2}\left(1 + (-1)^{s'+s''} e^{(\mu'-\mu'')/2\pi}\right)\delta(\mu'-\mu'')\,,$$
$$\forall\ \ \mu'\,,\,\mu''\in(0,\infty),\ \ \forall\ \ s',\,s''\in\{+,-\}\,. \quad (6.240)$$

So, if $(\mu',s')\neq(\mu'',s'')$, then the functions $u_{s'}(t,\mu')$ and $v_{s''}(t,\mu'')$ are orthogonal.

Let us calculate the "angle" between the *generalized* eigenfunctions $u_{s'}(t,\mu')$ and $u_{s'\prime}(t,\mu'')$. We recall that if x and y be non-zero vectors of some Hilbert space $\mathfrak{H}$, then the angle $\theta(x,y)$ between x and y is the unique $\theta\in[0,\pi/2]$ such that

$$\cos^2\theta = \frac{|\langle\, x\,,\,y\,\rangle|^2}{\langle\, x\,,\,x\,\rangle\,\langle\, y\,,\,y\,\rangle}\,. \quad (6.241)$$

Of course, the eigenfunctions $u_s(\mu)$ do not belong to the Hilbert space $L^2(0,\infty)$. We calculate the angle formally, interpreting ratios of δ-functions correspondently.

So, let us calculate the angle

$$\theta\big(u_{s'}(\,.\,,\mu')\,,u_{s''}(\,.\,,\mu''\,)\big)$$

between $x = u_{s'}(\,.\,,\mu')$ and $y = u_{s''}(\,.\,,\mu''\,)$.

First we consider the case $s'=s''(=s)$. According to (6.238),

$$|\langle\, x\,,\,y\,\rangle| = \frac{1}{2}\Big(1 + e^{(\mu'+\mu'')\pi/2}\Big)\,\delta(\mu'-\mu''),$$

and

$$\langle\, x\,,\, x\,\rangle = \frac{1}{2}\Big(1+e^{\mu'\pi}\Big)\,\delta(0), \quad \langle\, y\,,\, y\,\rangle = \frac{1}{2}\Big(1+e^{\mu''\pi}\Big)\delta(0)\,.$$

Thus, for $(x = u_s(\,.\,,\mu'),\, y = u_s(\,.\,,\mu''\,)$ the ratio in (6.241) is:

$$\frac{|\langle\, x\,,\, y\,\rangle|^2}{\langle\, x\,,\, x\,\rangle\,\langle\, y\,,\, y\,\rangle} = \frac{\cosh^2\frac{(\mu'+\mu'')\pi}{4}}{\cosh\frac{\mu'\pi}{2}\cdot\cosh\frac{\mu''\pi}{2}}\cdot\frac{\delta^2(\mu'-\mu'')}{\delta^2(0)}\,.$$

Interpreting the ratio $\dfrac{\delta^2(\mu'-\mu'')}{\delta^2(0)}$ *as zero if* $\mu' \neq \mu''$, *and as one if* $\mu' = \mu''$, we deduce that for every $s = \pm$ and for every $\mu',\, \mu'' \in (0,\infty)$,

$$\cos^2\theta\big(u_s(\,.\,,\mu')\,,\, u_s(\,.\,,\mu'')\big) = \begin{cases} 1 & \text{if } \ \mu' = \mu''\,, \\ 0 & \text{if } \ \mu' \neq \mu''\,. \end{cases} \tag{6.242}$$

Now we consider the case $s' \neq s''$, say $s' = s$, $s'' = -s$, where s is either plus, or minus. According to (6.238),

$$|\langle\, x\,,\, y\,\rangle| = \Big(e^{(\mu'+\mu'')/2\pi} - 1\Big)\,\delta(\mu'-\mu''),$$

and

$$\langle\, x\,,\, x\,\rangle = \Big(1+e^{\mu'\pi}\Big)\,\delta(0), \quad \langle\, y\,,\, y\,\rangle = \Big(1+e^{\mu''\pi}\Big)\delta(0)\,.$$

Thus, for $(x = u_s(\,.\,,\mu'),\, y = u_{-s}(\,.\,,\mu''\,)$ the ratio in (6.241) is:

$$\frac{|\langle\, x\,,\, y\,\rangle|^2}{\langle\, x\,,\, x\,\rangle\,\langle\, y\,,\, y\,\rangle} = \frac{\sinh^2\frac{(\mu'+\mu'')\pi}{4}}{\cosh\frac{\mu'\pi}{2}\cdot\cosh\frac{\mu''\pi}{2}}\cdot\frac{\delta^2(\mu'-\mu'')}{\delta^2(0)}\,.$$

As before, *interpreting the ratio* $\dfrac{\delta^2(\mu'-\mu'')}{\delta^2(0)}$ *as zero if* $\mu' \neq \mu''$, *and as one if* $\mu' = \mu''$, we deduce that for for every $\mu',\, \mu'' \in (0,\infty)$,

$$\cos^2\theta\big(u_+(\,.\,,\mu')\,,\, u_-(\,.\,,\mu'')\big) = \begin{cases} \tanh^2\frac{\mu\pi}{2} & \text{if } \ \mu' = \mu'' = \mu\,, \\ 0 & \text{if } \ \mu' \neq \mu''\,. \end{cases} \tag{6.243}$$

In particular, whatever $\mu \in (0,\infty)$, the eigenfunction $u_+(\,.\,,\mu)$ and $u_-(\,.\,,\mu)$ are not orthogonal which each other. In more detail,

$$\sin^2\theta(u_+(\,.\,,\mu)\,, u_-(\,.\,,\mu)\,) = \frac{1}{\cosh^2\frac{\mu\pi}{2}}\,.$$

Taking into account the equality (6.128), we can express the angle $\theta(u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)$ between eigenfunctions $u_+(t,\mu)$ and $u_-(t,\mu)$ in terms of the eigenvalues $\zeta_+(\mu)=\zeta_-(\mu),\ \zeta_-(\mu)=-\zeta(\mu)$:

$$\sin\theta(u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)\,)=\frac{2|\zeta(\mu)|^2}{\sqrt{1+2|\zeta(\mu)|^2}}\,, \tag{6.244}$$

Remark 6.14. If $\mathcal{C}$ is a contractive operator in a Hilbert space $\mathfrak{H}$ and e_1, e_2 are its eigenvectors corresponding to the eigenvalues ζ_1, ζ_2 respectively, then the angle $\theta(e_1,\,e_2)$ between the eigenvectors e_1 and e_2 admits the estimate from below:

$$\sin\theta(e_1,\,e_2)\geq\frac{|\zeta_1-\zeta_2|}{|1-\zeta_1\overline{\zeta_2}|}\,. \tag{6.245}$$

This estimate is precise: given a numbers ζ_1, ζ_2: $|\zeta_1|<1$, $|\zeta_2|<1|$, there exists a contraction $\mathcal{C}$ in a Hilbert space such that the numbers ζ_1, ζ_2 are eigenvalues of $\mathcal{C}$ and for the angle between the appropriate eigenvectors the equality holds in (6.245).

In particular, if $\zeta_1=\zeta$ and $\zeta_2=-\zeta$, where $|\zeta|<1$, then the inequality (6.245) takes the form

$$\sin\theta(e_1,\,e_2)\geq\frac{2|\zeta|}{1+|\zeta|^2}\,. \tag{6.246}$$

The eigenfunctions $u_+(\,.\,,\mu),u_-(\,.\,,\mu)$ of the contractive operator $\mathcal{F}_E$, corresponding to the eigenvalues $\zeta(\mu)$ and $-\zeta(\mu)$, are generalized eigenfunctions rather than "conventional" one. Nevertheless, since

$$\frac{2|\zeta|}{\sqrt{1+2|\zeta|^2}}>\frac{2|\zeta|}{1+|\zeta|^2}\,, \tag{6.247}$$

the "angle" $\theta(u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)\,)$ between these generalized eigenfunction, which expressed by the equality (6.244), satisfy the estimate (6.246):

$$\sin\theta(u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)\,)>\frac{2|\zeta|}{1+|\zeta|^2}\,. \tag{6.248}$$

Moreover, the estimate (6.248) is almost precise for small ζ. The difference between the left- and the right hand sides in (6.248) is very small for small ζ:

$$0<\sin\theta(u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)\,)-\frac{2|\zeta(\mu)|}{1+|\zeta(\mu)|^2}<|\zeta(\mu)|^5,\ \ \forall\,\mu\in(0,\infty). \tag{6.249}$$

Of course the inequality (6.248) for the angle between the generalized eigenfunctions $u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)$ of the operator $\mathcal{F}_E$ is a track of the appropriate inequality (**??**) for the angle between eigenvectors (conventional) of the 2×2 matrix $F(\mu)$.

9. Let us justify the equality (6.244) for the angle between the *generalized* eigenfunctions $u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)$. We interpret this angle as an angle between (conventional) subspaces of the Hilbert space $\mathfrak{H}=L^2(0,\infty)$. We choose these subspaces in such a way that the pair $u_+(\,.\,,\mu)\,,u_-(\,.\,,\mu)$ of generalized functions appear as the weak limits of sequences of pairs of conventional functions on which the angle between the subspaces is asymptotically attained.

Definition 6.19. *Let $\mathfrak{H}$ be a Hilbert space, and $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of $\mathfrak{H}$, $\mathfrak{M}\neq\{0\}$, $\mathfrak{N}\neq\{0\}$. The angle $\theta(\mathfrak{M}\,,\,\mathfrak{N})$ is defined as*

$$\theta(\mathfrak{M}\,,\,\mathfrak{N})\stackrel{\text{def}}{=}\inf\theta(\mathfrak{m}\,,\,\mathfrak{n})\,,\tag{6.250}$$

where $\mathfrak{m}$, $\mathfrak{n}$ are non-zero vectors, $\mathfrak{m}\in\mathfrak{M}$ and $\mathfrak{n}\in\mathfrak{N}$, $\theta(\mathfrak{m}\,,\,\mathfrak{n})$ is the angle between $\mathfrak{m}$ and $\mathfrak{n}$, and inf *is taken over all non-zero $\mathfrak{m}\in\mathfrak{M}$, $\mathfrak{n}\in\mathfrak{N}$. In other words,*

$$0\leq\theta(\mathfrak{M}\,,\,\mathfrak{N})\leq\pi/2\,,\quad\cos^2\theta(\mathfrak{M}\,,\,\mathfrak{N})=\sup\frac{|\langle\,\mathfrak{m}\,,\,\mathfrak{n}\,\rangle|^2}{\langle\,\mathfrak{m}\,,\,\mathfrak{m}\,\rangle\,\langle\,\mathfrak{n}\,,\,\mathfrak{n}\,\rangle}\,,$$

and sup *is taken over all non-zero $\mathfrak{m}\in\mathfrak{M}$, $\mathfrak{n}\in\mathfrak{N}$.*
(*Without loss of generality the* inf *may be taken only over all normalized vectors $\mathfrak{m}$ and $\mathfrak{n}$: that is $\|\mathfrak{m}\|=1$, $\|\mathfrak{n}\|=1$.*)

Remark 6.15. *If the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ have non-trivial intersection, that is $\mathfrak{M}\cap\mathfrak{N}\neq\{0\}$, where $\{0\}$ is the zero vector, then $\theta(\mathfrak{M}\,,\,\mathfrak{N})=0$. If the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ are closed, at least one of them is of finite dimension: $\min\big(\dim(\mathfrak{M}),\,\dim(\mathfrak{N})\big)<\infty$ and these subspaces have the trivial intersection: $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, then $\theta(\mathfrak{M}\,,\,\mathfrak{N})>0$. However, if both the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ are infinite-dimensional, then it may happen that $\mathfrak{M}$ is closed, $\mathfrak{N}$ is closed, $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, but nevertheless $\theta(\mathfrak{M}\,,\,\mathfrak{N})=0$: the* inf *in* (6.250) *equals zero, but there is no non-zero vectors $\mathfrak{m}$, $\mathfrak{n}$ on which the* inf *is attained.*

Definition 6.20. *If $\mathfrak{M}$ and $\mathfrak{N}$ be subspaces of $\mathfrak{H}$ (closed or not), then we can consider their sum $\mathfrak{M}+\mathfrak{N}$. If moreover the intersection of these subspaces is trivial: $\mathfrak{M}\cap\mathfrak{N}=\{0\}$, then the sum $\mathfrak{M}+\mathfrak{N}$ is* direct*: $\mathfrak{M}+\mathfrak{N}=\mathfrak{M}\dot{+}\mathfrak{N}$: every vector $\mathfrak{x}\in\mathfrak{M}+\mathfrak{N}$ has* unique *representation of*

the form $\mathfrak{x} = \mathfrak{m} + \mathfrak{n}$*, where* $\mathfrak{m} \in \mathfrak{M}$, $\mathfrak{n} \in \mathfrak{N}$*. In this case, two projector operators are defined:* $P_{\mathfrak{M}\|\mathfrak{N}}$ *and* $P_{\mathfrak{N}\|\mathfrak{M}}$*:*

$$
\begin{gathered}
P_{\mathfrak{M}\|\mathfrak{N}}: \ \mathfrak{M} \dotplus \mathfrak{N} \to \mathfrak{M}, \quad P_{\mathfrak{M}\|\mathfrak{N}}: \ \mathfrak{M} \dotplus \mathfrak{N} \to \mathfrak{N}, \\
\text{for } \mathfrak{x} = \mathfrak{m} + \mathfrak{n}, \ \text{where } \mathfrak{m} \in \mathfrak{M}, \, \mathfrak{n} \in \mathfrak{N}, \quad P_{\mathfrak{M}\|\mathfrak{N}}\, \mathfrak{x} = \mathfrak{m}, \quad P_{\mathfrak{M}\|\mathfrak{N}}\, \mathfrak{x} = \mathfrak{n}\,.
\end{gathered}
\tag{6.251}
$$

The operators $P_{\mathfrak{M}\|\mathfrak{N}}$ *and* $P_{\mathfrak{N}\|\mathfrak{M}}$*, which are projector operators, are said to be* the projector onto $\mathfrak{M}$ parallel to $\mathfrak{N}$ *and* the projector onto $\mathfrak{N}$ parallel to $\mathfrak{M}$ *respectively.*

Lemma 6.17. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be subspaces of a Hilbert space* $\mathfrak{H}$*, and* $\mathfrak{M} \cap \mathfrak{N} = \{0\}$*. Then the angle* $\theta\big(\mathfrak{M}, \mathfrak{N}\big)$ *and the norm of the projector* $P_{\mathfrak{M}\|\mathfrak{N}}$ *are related by the equality*

$$
\frac{1}{\sin\theta\big(\mathfrak{M}, \mathfrak{N}\big)} = \big\|P_{\mathfrak{M}\|\mathfrak{N}}\big\|\,. \tag{6.252}
$$

This lemma is a standard fact of the geometry of Hilbert space.

In what follows we use the following terminology: *two subspaces, say* $\mathfrak{K}$ *and* $\mathfrak{L}$*, of a Hilbert space* $\mathfrak{H}$ *are* orthogonal *if the scalar product* $\langle\, \mathfrak{k}\,, \mathfrak{l}\,\rangle = 0$ *for every* $\mathfrak{k} \in \mathfrak{K}\,, \mathfrak{l} \in \mathfrak{L}$*.* We denote that the subspaces $\mathfrak{K}$ and $\mathfrak{L}$ are orthogonal by $\mathfrak{K} \perp \mathfrak{L}$.

Lemma 6.18. *Let* $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{N}_1$, $\mathfrak{N}_2$*, be subspaces of a Hilbert space* $\mathfrak{H}$*, and* $\mathfrak{M}_1 \neq \{0\}$, $\mathfrak{N}_1 \neq \{0\}$*. Assume that the orthogonality condition*

$$
\big(\mathfrak{M}_2 + \mathfrak{N}_2\big) \perp \big(\mathfrak{M}_1 + \mathfrak{N}_1\big) \tag{6.253}
$$

is satisfied. Then

1. *If* $\mathfrak{M}_2 \neq \{0\}$, $\mathfrak{N}_2 \neq 0$*, the angle* $\theta(\mathfrak{M}_1 \oplus \mathfrak{M}_2\,, \mathfrak{N}_1 \oplus \mathfrak{N}_2)$ *between the subspaces* $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ *and* $\mathfrak{N}_1 \oplus \mathfrak{N}_2$ *is:*

$$
\theta(\mathfrak{M}_1 \oplus \mathfrak{M}_2\,, \mathfrak{N}_1 \oplus \mathfrak{N}_1) = \min\big(\theta(\mathfrak{M}_1, \mathfrak{N}_1)\,, \theta(\mathfrak{M}_2, \mathfrak{N}_2)\big)\,. \tag{6.254}
$$

   In particular, if $\mathfrak{M}_2 \perp \mathfrak{N}_2 = 0$*, then*

$$
\theta(\mathfrak{M}_1 \oplus \mathfrak{M}_2\,, \mathfrak{N}_1 \oplus \mathfrak{N}_2) = \theta(\mathfrak{M}_1, \mathfrak{N}_1)\,. \tag{6.255}
$$

2. *If one of the subspaces* $\mathfrak{M}_2$, $\mathfrak{N}_2$ *is trivial, that is either* $\mathfrak{M}_2 = \{0\}$*, or* $\mathfrak{N}_2 = \{0\}$*, then the equality* (6.255) *holds as well.* (*In this case, the angle* $\theta(\mathfrak{M}_2, \mathfrak{N}_2)$ *is not defined.*)

Proof. Let $\mathfrak{m}_1 \in \mathfrak{M}_1$, $\mathfrak{m}_2 \in \mathfrak{M}_2$, $\mathfrak{n}_1 \in \mathfrak{N}_1$, $\mathfrak{n}_2 \in \mathfrak{N}_2$, and $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$, $n = n_1 + n_2$. Due to the conditions (6.253),

$$\langle\, \mathfrak{m}\,,\, \mathfrak{n}\,\rangle = \langle\, \mathfrak{m}_1\,,\, \mathfrak{n}_1\,\rangle + \langle\, \mathfrak{m}_2\,,\, \mathfrak{n}_2\,\rangle\,, \tag{6.256a}$$

$$\|\mathfrak{m}\|^2 = \|\mathfrak{m}_1\|^2 + \|\mathfrak{m}_2\|^2, \quad \|\mathfrak{n}\|^2 = \|\mathfrak{n}_1\|^2 + \|\mathfrak{n}_2\|^2\,. \tag{6.256b}$$

Since $|\langle\, \mathfrak{m}_j\,,\, \mathfrak{n}_j\,\rangle| \leq \cos\theta(\mathfrak{M}_j,\, \mathfrak{N}_j)\|\mathfrak{m}_j\|\,\|\mathfrak{n}_j\|,\ j = 1,\, 2$, the inequality

$$\big|\langle\, \mathfrak{m}\,,\, \mathfrak{n}\,\rangle\big| \leq \cos\theta(\mathfrak{M}_1,\, \mathfrak{N}_1)\|\mathfrak{m}_1\|\,\|\mathfrak{n}_1\| + \cos\theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\|\mathfrak{m}_2\|\,\|\mathfrak{n}_2\|\,,$$

holds. Hence, the inequality

$$\begin{aligned}\big|\langle\, \mathfrak{m}\,,\, \mathfrak{n}\,\rangle\big|^2 \leq \cos^2\big(\min(\theta(\mathfrak{M}_1,\, \mathfrak{N}_1),\, \theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\big)\,\cdot \\ \cdot\big(\|\mathfrak{m}_1\|^2 + \|\mathfrak{m}_2\|^2\big)\cdot\big(\|\mathfrak{m}_1\|^2 + \|\mathfrak{m}_2\|^2\big)\end{aligned}$$

holds. According to (6.256b), this inequality takes the form

$$\big|\langle\, \mathfrak{m}\,,\, \mathfrak{n}\,\rangle\big|^2 \leq \cos^2\big(\min(\theta(\mathfrak{M}_1,\, \mathfrak{N}_1),\, \theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\big)\,\cdot\,\|\mathfrak{m}\|^2\cdot\|\mathfrak{n}\|^2,$$

that is

$$\cos^2\theta(\mathfrak{m},\, \mathfrak{n}) \leq \cos^2\big(\min(\theta(\mathfrak{M}_1,\, \mathfrak{N}_1),\, \theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\big)$$

for every non-zero $\mathfrak{m} \in \mathfrak{M}$, $\mathfrak{n} \in \mathfrak{N}$. Thus,

$$\cos^2\theta(\mathfrak{M},\, \mathfrak{N}) \leq \cos^2\big(\min(\theta(\mathfrak{M}_1,\, \mathfrak{N}_1),\, \theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\big),$$

or

$$\theta(\mathfrak{M},\, \mathfrak{N}) \geq \min\big(\theta(\mathfrak{M}_1,\, \mathfrak{N}_1),\, \theta(\mathfrak{M}_2,\, \mathfrak{N}_2)\big)\,.$$

The inverse inequality is evident.

This reasoning is also applicable if either $\mathfrak{M}_2 = \{0\}$, or $\mathfrak{N}_2 = \{0\}$. $\square$

We apply Lemma 6.18 to study the angles between the spectral subspaces $\mathcal{H}_{\mathfrak{F}_E}(\Delta)$. (See Definitions 6.17 and 6.18.)

Lemma 6.19. *Let Δ, $\Delta \subseteq \sigma_{\mathfrak{F}_E}$, be an admissible set which is asymmetric, that is $\Delta \cap (-\Delta) = \emptyset_e$.*

Then

$$\sin\theta(\mathcal{H}_{\mathfrak{F}_E}(\Delta),\, \mathcal{H}_{\mathfrak{F}_E}(-\Delta)) = \frac{2d}{\sqrt{1+2d^2}}\,, \quad \text{where} \quad d = \operatorname{ess\,dist}(\Delta\,,\, 0). \tag{6.257}$$

Remark 6.16. *If* mes $(\Delta\cap(-\Delta))\neq 0$*, then the subspaces* $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ *and* $\mathcal{H}_{\mathcal{F}_E}(-\Delta)$ *have non-trivial intersection:* $\mathcal{H}_{\mathcal{F}_E}(\Delta)\cap\mathcal{H}_{\mathcal{F}_E}(-\Delta)\neq\{0\}$*, thus* $\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta),\,\mathcal{H}_{\mathcal{F}_E}(-\Delta))=0$.

Proof of Lemma 6.19. In view of (6.190) and (6.189), the spectral projector $\mathcal{P}_{\mathcal{F}_E}(\Delta)$, defined on the whole space $L^2(\mathbb{R}_+)$, is the projector onto the subspace $\mathcal{H}_{\mathcal{F}_E}(\Delta)$ parallel to the subspace $\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta)$. According to Lemma 6.17,

$$\sin\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta))=\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\|^{-1}\,.$$

From the other hand, according to Lemma 6.12, statement 2,

$$\|\mathcal{P}_{\mathcal{F}_E}(\Delta)\|=\frac{\sqrt{1+2d^2}}{2d}\,.$$

Thus,

$$\sin\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta))=\frac{2d}{\sqrt{1+2d^2}}\,.$$

We apply now Lemma 6.18 to prove that

$$\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(-\Delta))=\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\Delta))\,. \tag{6.258}$$

We set

$$\begin{aligned}&\mathfrak{M}_1=\mathcal{H}_{\mathcal{F}_E}(\Delta),\quad \mathfrak{N}_1=\mathcal{H}_{\mathcal{F}_E}(-\Delta),\\ &\qquad\qquad\qquad \mathfrak{M}_2=\{0\},\quad \mathfrak{N}_2=\mathcal{H}_{\mathcal{F}_E}(\sigma_{\mathcal{F}_E}\setminus\big(\Delta\cup(-\Delta)\big))\,.\end{aligned}$$

The sets $\Delta\cup(-\Delta)$ and $\sigma_{\mathcal{F}_E}\setminus\big(\Delta\cup(-\Delta)\big)$ are symmetric and do not intersect. According to Theorem 6.12, the subspace $\mathfrak{N}_2$ is orthogonal to the subspace $\mathfrak{M}_1+\mathfrak{N}_1$. By Lemma 6.18,

$$\begin{aligned}&\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(-\Delta))=\\ &\qquad\qquad=\theta\Big(\mathcal{H}_{\mathcal{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathcal{F}_E}(-\Delta)\oplus\mathcal{H}_{\mathcal{F}_E}\big(\sigma_{\mathcal{F}_E}\setminus\big(\Delta\cup(-\Delta)\big)\big)\Big)\end{aligned}$$

From the other hand, since $(-\Delta)\cup\big(\sigma_{\mathcal{F}_E}\setminus\big(\Delta\cup(-\Delta)\big)\big)=\sigma_{\mathcal{F}_E}\setminus\Delta$,

$$\mathcal{H}_{\mathcal{F}_E}(-\Delta)\oplus\mathcal{H}_{\mathcal{F}_E}\big(\sigma_{\mathcal{F}_E}\setminus\big(\Delta\cup(-\Delta)\big)\big)=\mathcal{H}_{\mathcal{F}_E}\big(\sigma_{\mathcal{F}_E}\setminus\Delta\big)\,.$$

So, the equality (6.258) holds. $\square$

Theorem 6.15. *Let Δ_1, Δ_2, $\Delta_1 \subseteq \sigma_{\mathcal{F}_E}$, $\Delta_2 \subseteq \sigma_{\mathcal{F}_E}$, be admissible sets, and*

$$\Delta_1 \cap \Delta_2 = \emptyset_e. \tag{6.259}$$

Let

$$\Delta \stackrel{\text{def}}{=} \Delta_1 \cap (-\Delta_2)\,. \tag{6.260}$$

1. *If $\Delta = \emptyset_e$, then the spectral subspaces $\mathcal{H}_{\mathcal{F}_E}(\Delta_1)$ and $\mathcal{H}_{\mathcal{F}_E}(\Delta_2)$ are mutually orthogonal: i.e.*

$$\sin\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta_1)\,,\,\mathcal{H}_{\mathcal{F}_E}(\Delta_2)) = 1\,.$$

2. *If $\Delta \neq \emptyset_e$, then the spectral subspaces $\mathcal{H}_{\mathcal{F}_E}(\Delta_1)$ and $\mathcal{H}_{\mathcal{F}_E}(\Delta_2)$ are not orthogonal, and*

$$\sin\theta(\mathcal{H}_{\mathcal{F}_E}(\Delta_1), \mathcal{H}_{\mathcal{F}_E}(\Delta_2)) = \frac{2d}{\sqrt{1+2d^2}}, \text{ where } d{=}\operatorname{ess\,dist}(\Delta, 0)\,.$$

In particular, if the condition (6.259) *holds and either $\Delta_1 \subseteq \sigma^+_{\mathcal{F}_E}$, $\Delta_2 \subseteq \sigma^+_{\mathcal{F}_E}$, or $\Delta_1 \subseteq \sigma^-_{\mathcal{F}_E}$, $\Delta_2 \subseteq \sigma^-_{\mathcal{F}_E}$, then the subspaces $\mathcal{H}_{\mathcal{F}_E}(\Delta_1)$ and $\mathcal{H}_{\mathcal{F}_E}(\Delta_2)$ are mutually orthogonal.*

Proof of Theorem 6.15. The proof is based on Theorem 6.12 and on Lemmas 6.18, 6.19. Let us set

$$\Delta_1^{(r)} = \Delta_1 \setminus \Delta, \quad \Delta_2^{(r)} = \Delta_2 \setminus (-\Delta)\,.$$

Keeping in mind to use Lemma 6.18, we denote

$$\begin{aligned} \mathfrak{M}_1 &= \mathcal{H}_{\mathcal{F}_E}(\Delta)\,, & \mathfrak{N}_1 &= \mathcal{H}_{\mathcal{F}_E}(-\Delta)\,, \\ \mathfrak{M}_2 &= \mathcal{H}_{\mathcal{F}_E}(\Delta_1^{(r)}), & \mathfrak{N}_2 &= \mathcal{H}_{\mathcal{F}_E}(\Delta_2^{(r)})\,. \end{aligned}$$

It is clear that $\Delta \cap \Delta_1^{(r)} = \emptyset$. Moreover, using (6.260), we see that

$$(-\Delta) \cap \Delta_1^{(r)} = (-\Delta_1) \cap \Delta_2 \cap (\Delta_1 \setminus \Delta) \subseteq \Delta_1 \cap \Delta_2 = \emptyset_e\,.$$

Thus,

$$(\Delta \cup (-\Delta)) \cap \Delta_1^{(r)} = \emptyset_e, \quad (\Delta \cup (-\Delta)) \cap (-\Delta_1^{(r)}) = \emptyset_e\,.$$

By Theorem 6.12, $(\mathfrak{M}_1 + \mathfrak{N}_1) \perp \mathfrak{M}_2$. Analogously, $(\mathfrak{M}_1 + \mathfrak{N}_1) \perp \mathfrak{N}_2$. Thus the condition (6.253) holds.

Let us show that $\mathfrak{M}_2 \perp \mathfrak{N}_2$. Since $\Delta_1^{(r)} \subseteq \Delta_1$, $\Delta_2^{(r)} \subseteq \Delta_2$, and (6.259), we have $\Delta_1^{(r)} \cap \Delta_2^{(r)} = \emptyset_e$. Since $\Delta_1^{(r)} \subseteq \Delta_1$ and $-\Delta_2^{(r)} = (-\Delta_2) \setminus \Delta_1$, we have $\Delta_1^{(r)} \cap (-\Delta_2^{(r)}) = \emptyset_e$. Thus,

$$\Delta_1^{(r)} \cap (\Delta_2^{(r)} \cup (-\Delta_2^{(r)})) = \emptyset_e\,, \quad -\Delta_1^{(r)} \cap (\Delta_2^{(r)} \cup (-\Delta_2^{(r)})) = \emptyset_e\,.$$

By Theorem (6.12), $\mathfrak{M}_2 \perp \mathfrak{N}_2$.

By Lemma 6.18,

$$\theta(\mathfrak{M}_1,\ \mathfrak{N}_1) = \theta\big(\mathfrak{M}_1 \oplus \mathfrak{M}_2\,, \mathfrak{N}_1 \oplus \mathfrak{N}_2\big)\,.$$

Since $\Delta \cup (\Delta_1^{(r)}) = \Delta_1$, then $\mathfrak{M}_1 \oplus \mathfrak{M}_2 = \mathcal{H}_{\mathfrak{F}_E}(\Delta_1)$. Analogously, $\mathfrak{N}_1 \oplus \mathfrak{N}_2 = \mathcal{H}_{\mathfrak{F}_E}(\Delta_2)$. Thus,

$$\theta\big(\mathcal{H}_{\mathfrak{F}_E}(\Delta_1)\,, \mathcal{H}_{\mathfrak{F}_E}(\Delta_2)\big) = \theta\big(\mathcal{H}_{\mathfrak{F}_E}(\Delta)\,, \mathcal{H}_{\mathfrak{F}_E}(-\Delta)\big). \tag{6.261}$$

The angle in the right hand side of the last equality was calculated in Lemma 6.19. $\square$

Lemma 6.20. *Let Δ, $\Delta \subseteq \sigma_{\mathfrak{F}_E}$, be an admissible set which is asymmetric, that is $\Delta \cap (-\Delta) = \emptyset_e$. For $\varepsilon > 0$, set*

$$\Delta_\varepsilon = \Delta \cap \{z \in \mathbb{C} :\ |z| \leq d+\varepsilon\}\,,\ (-\Delta)_\varepsilon = (-\Delta) \cap \{z \in \mathbb{C} :\ |z| \leq d+\varepsilon\}\,, \tag{6.262}$$

where d is the same that in (6.257).

Then for arbitrary $\varepsilon > 0$*,*

$$\theta(\mathcal{H}_{\mathfrak{F}_E}(\Delta_\varepsilon)\,, \mathcal{H}_{\mathfrak{F}_E}((-\Delta)_\varepsilon) = \theta(\mathcal{H}_{\mathfrak{F}_E}(\Delta)\,, \mathcal{H}_{\mathfrak{F}_E}(-\Delta)\,, \tag{6.263}$$

In other words, only the arbitrary small *'partial' subspaces $\mathcal{H}_{\mathfrak{F}_E}(\Delta_\varepsilon)$ and $\mathcal{H}_{\mathfrak{F}_E}((-\Delta)_\varepsilon)$ are responsible for the angle between the 'whole' subspaces $\mathcal{H}_{\mathfrak{F}_E}(\Delta)$ and $\mathcal{H}_{\mathfrak{F}_E}(-\Delta)$ rather the 'whole' subspaces themselves.*

Proof. The proof is based on Theorem 6.12 and on Lemmas 6.18, 6.19. Let us set

$$\begin{aligned} \mathfrak{M}_1 &= \mathcal{H}_{\mathfrak{F}_E}(\Delta_\varepsilon)\,, & \mathfrak{N}_1 &= \mathcal{H}_{\mathfrak{F}_E}((-\Delta)_\varepsilon)\,, \\ \mathfrak{M}_2 &= \mathcal{H}_{\mathfrak{F}_E}(\Delta \setminus \Delta_\varepsilon), & \mathfrak{N}_2 &= \mathcal{H}_{\mathfrak{F}_E}((-\Delta) \setminus (-\Delta)_\varepsilon)\,. \end{aligned}$$

We assume that $\Delta \setminus \Delta_\varepsilon \neq \emptyset_\varepsilon$, in other case the assertion of Lemma 6.20 is evident. By Lemma 6.19,

$$\sin\theta(\mathfrak{M}_1\,, \mathfrak{N}_1) < \sin\theta(\mathfrak{M}_2\,, \mathfrak{N}_2). \tag{6.264}$$

Indeed, $d_1 < d_2$, where $d_1 \stackrel{\text{def}}{=} \operatorname{ess\,dist}(\Delta_\varepsilon,\, 0)$, $d_2 \stackrel{\text{def}}{=} \operatorname{ess\,dist}(\Delta \setminus \Delta_\varepsilon,\, 0)$. ($d_1$ is the same, that d. Actually, $d_1 + \varepsilon \leq d_2$.) According to Lemma 6.19, $\sin\theta(\mathfrak{M}_1\,,\,\mathfrak{N}_1) = \frac{2d_1}{\sqrt{1+2d_1^2}}$, $\sin\theta(\mathfrak{M}_2\,,\,\mathfrak{N}_2) = \frac{2d_2}{\sqrt{1+2d_2^2}}$. Since $\frac{d_1}{\sqrt{1+2d_1^2}} < \frac{d_2}{\sqrt{1+2d_2^2}}$ if $d_1 < d_2$, the inequality (6.264) holds.

To apply Lemma 6.18, we have to check that $(\mathfrak{M}_1 + \mathfrak{N}_1) \perp (\mathfrak{M}_2 + \mathfrak{N}_2)$. This follows from Theorem 6.12 if we check that

$$\big(\Delta_\varepsilon \cup (-\Delta_\varepsilon)\big) \cap \big(\Delta \setminus \Delta_\varepsilon\big) = \emptyset_e\,.$$

This is true: the equality $\Delta_\varepsilon \cap \big(\Delta \setminus \Delta_\varepsilon\big) = \emptyset_e$ is evident, the equality $(-\Delta_\varepsilon) \cap \big(\Delta \setminus \Delta_\varepsilon\big) = \emptyset_e$ holds since $(-\Delta_\varepsilon) \cap \big(\Delta \setminus \Delta_\varepsilon\big) \subseteq (-\Delta) \cap \Delta$. $\square$

The following result supplements Lemma 6.20:

Theorem 6.16. *Let Δ, $\Delta \subseteq \sigma_{\mathfrak{F}_E}$, be an admissible set which is asymmetric, that is $\Delta \cap (-\Delta) = \emptyset_e$. Let $\{x_n\}_{1 \leq n < \infty}$, $\{y_n\}_{1 \leq n < \infty}$ be sequences of vectors on which the angle $\theta\big(\mathcal{H}_{\mathfrak{F}_E}(\Delta)\,,\,\mathcal{H}_{\mathfrak{F}_E}(-\Delta)\big)$ between the spectral subspaces $\mathcal{H}_{\mathfrak{F}_E}(\Delta)$ and $\mathcal{H}_{\mathfrak{F}_E}(-\Delta)$ is attained, that is*

$$x_n \in \mathcal{H}_{\mathfrak{F}_E}(\Delta),\ x_n \neq 0, \quad y_n \in \mathcal{H}_{\mathfrak{F}_E}(-\Delta),\ x_n \neq 0 \ \text{ for every } \quad n\,,$$

and

$$\lim_{n\to\infty} \cos\theta(x_n,\, y_n) = \cos\theta(\mathcal{H}_{\mathfrak{F}_E}(\Delta),\, \mathcal{H}_{\mathfrak{F}_E}(-\Delta))\,.$$

For $\varepsilon > 0$, let $\Delta = \Delta_1(\varepsilon) \cup \Delta_2(\varepsilon)$ be the partition of the set Δ, where

$$\Delta_1(\varepsilon) = \Delta \cap \langle z \in \mathbb{C} :\ |z| < d + \varepsilon\rangle,\ \Delta_2(\varepsilon) = \Delta \cap \langle z \in \mathbb{C} :\ z| \geq d + \varepsilon\rangle,$$

$d = \operatorname{ess\,dist}(\Delta,\, 0)$*, and*

$$x_n = x_n^{(1)}(\varepsilon) \oplus x_n^{(2)}(\varepsilon),\ \ y_n = y_n^{(1)}(\varepsilon) \oplus y_n^{(2)}(\varepsilon)$$

be the corresponding decompositions of the vectors x_n and y_n:

$$x_n^{(j)} \in \mathcal{H}_{\mathfrak{F}_E}(\Delta_j(\varepsilon)),\ \ y_n^{(j)} \in \mathcal{H}_{\mathfrak{F}_E}(-\Delta_j(\varepsilon)), \quad j = 1,\, 2.$$

Then for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\|x_n^{(2)}(\varepsilon)\|}{\|x_n\|} = 0, \quad \lim_{n\to\infty} \frac{\|y_n^{(2)}(\varepsilon)\|}{\|y_n\|} = 0$$

In other words, the spectra of the vectors x_n and y_n are concentrating towards the two-point set $\{-de^{i\pi/4},\, de^{i\pi/4}\}$ as $n \to \infty$.

Proof. The reasoning used in the proof of Lemma 6.20 showns that

$$\big(\mathcal{H}_{\mathcal{F}_E}(\Delta_1(\varepsilon)) + \mathcal{H}_{\mathcal{F}_E}(-\Delta_1(\varepsilon))\big) \perp \big(\mathcal{H}_{\mathcal{F}_E}(\Delta_2(\varepsilon)) + \mathcal{H}_{\mathcal{F}_E}(-\Delta_2(\varepsilon))\big)\,.$$

Therefore

$$\langle x_n\,,\, y_n\rangle = \langle x_n^{(1)}(\varepsilon)\,,\, y_n^{(1)}(\varepsilon)\rangle \;+\; \langle x_n^{(2)}(\varepsilon)\,,\, y_n^{(2)}(\varepsilon)\rangle\,, \tag{6.265a}$$
$$\langle x_n\,,\, x_n\rangle = \langle x_n^{(1)}(\varepsilon)\,,\, x_n^{(1)}(\varepsilon)\rangle \;+\; \langle x_n^{(2)}(\varepsilon)\,,\, x_n^{(2)}(\varepsilon)\rangle\,, \tag{6.265b}$$
$$\langle y_n\,,\, y_n\rangle = \langle y_n^{(1)}(\varepsilon)\,,\, y_n^{(1)}(\varepsilon)\rangle \;+\; \langle y_n^{(2)}(\varepsilon)\,,\, y_n^{(2)}(\varepsilon)\rangle\,. \tag{6.265c}$$

From (6.265a) it follows that

$$|\langle x_n\,,\, y_n\rangle| \leq |\langle x_n^{(1)}(\varepsilon)\,,\, y_n^{(1)}(\varepsilon)\rangle| + |\langle x_n^{(2)}(\varepsilon)\,,\, y_n^{(2)}(\varepsilon)\rangle|\,,$$

thus

$$|\langle x_n\,,\, y_n\rangle| \leq \cos\theta_1 \cdot \|x_n^{(1)}(\varepsilon)\| \cdot \|y_n^{(1)}(\varepsilon)\| + \cos\theta_2 \cdot \|x_n^{(2)}(\varepsilon)\| \cdot \|y_n^{(2)}(\varepsilon)\|\,, \tag{6.266}$$

where

$$\theta_j = \theta(\mathcal{H}_{\mathcal{F}_E}(\Delta_j(\varepsilon))\,,\, \mathcal{H}_{\mathcal{F}_E}(\Delta_j(-\varepsilon)))\,, \quad j = 1\,,2\,.$$

Using the Cauchy inequality, we derive from (6.266) that

$$\frac{|\langle x_n\,,\, y_n\rangle|^2}{\|x_n\|^2\,\|y_n\|^2} \leq \left(\cos\theta_1\,\frac{\|x_n^{(1)}(\varepsilon)\|^2}{\|x_n\|^2} + \cos\theta_2\,\frac{\|x_n^{(2)}(\varepsilon)\|^2}{\|x_n\|^2}\right)\cdot$$
$$\cdot\left(\cos\theta_1\,\frac{\|y_n^{(1)}(\varepsilon)\|^2}{\|y_n\|^2} + \cos\theta_2\,\frac{\|y_n^{(2)}(\varepsilon)\|^2}{\|y_n\|^2}\right).$$

Since $0 < \operatorname{ess\,dist}(\Delta_1(\varepsilon,\, 0)) = d < d+\varepsilon \leq \operatorname{ess\,dist}(\Delta_2(\varepsilon,\, 0))$, the *strict* inequality

$$\cos\theta_1 > \cos\theta_2 \tag{6.267}$$

holds. From (6.267) and (6.265c) it follows that

$$\cos\theta_1\,\frac{\|y_n^{(1)}(\varepsilon)\|^2}{\|y_n\|^2} + \cos\theta_2\,\frac{\|y_n^{(2)}(\varepsilon)\|^2}{\|y_n\|^2} \leq \cos\theta_1,$$

thus the inequality

$$\frac{|\langle x_n\,,\, y_n\rangle|^2}{\|x_n\|^2\,\|y_n\|^2} \leq \cos\theta_1\left(\cos\theta_1\,\frac{\|x_n^{(1)}(\varepsilon)\|^2}{\|x_n\|^2} + \cos\theta_2\,\frac{\|x_n^{(2)}(\varepsilon)\|^2}{\|x_n\|^2}\right)$$

holds. If the condition $\lim_{n\to\infty} \frac{\|x_n^{(2)}(\varepsilon)\|}{\|x_n\|} = 0$ is violated, then for some α, $0 < \alpha < 1$, and for infinitely many n, the inequality

$$\frac{\|x_n^{(2)}(\varepsilon)\|}{\|x_n\|} \geq \alpha \tag{6.268}$$

holds. Passing to subsequence, we assume that the last inequality holds for all n. From (6.267) it follows that the function $(1-t)\cos\theta_1 + t\cos\theta_2$ decreases in t. From (6.265b) and (6.268) it follows that

$$\cos\theta_1 \frac{\|x_n^{(1)}(\varepsilon)\|^2}{\|x_n\|^2} + \cos\theta_2 \frac{\|x_n^{(2)}(\varepsilon)\|^2}{\|x_n\|^2} \leq (1-\alpha)\cos\theta_1 + \alpha\cos\theta_2\,.$$

Therefore the inequality

$$\frac{|\langle x_n\,,\, y_n\rangle|^2}{\|x_n\|^2\,\|y_n\|^2} \leq \cos\theta_1 \cdot \big((1-\alpha)\cos\theta_1 + \alpha\cos\theta_2\big)$$

holds for all n.

Taking into account that $\lim_{n\to\infty} \dfrac{|\langle x_n\,,\, y_n\rangle|^2}{\|x_n\|^2\,\|y_n\|^2} = \cos^2\theta_1$, we conclude that the inequality

$$\cos\theta_1 \leq (1-\alpha)\cos\theta_1 + \alpha\cos\theta_2\,.$$

holds for some α, $0 < \alpha < 1$. However, this inequality contradicts the strict inequality (6.267). So we proved that $\lim_{n\to\infty} \frac{\|x_n^{(2)}(\varepsilon)\|}{\|x_n\|} = 0$. Analogously, $\lim_{n\to\infty} \frac{\|y_n^{(2)}(\varepsilon)\|}{\|y_n\|} = 0$. □

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